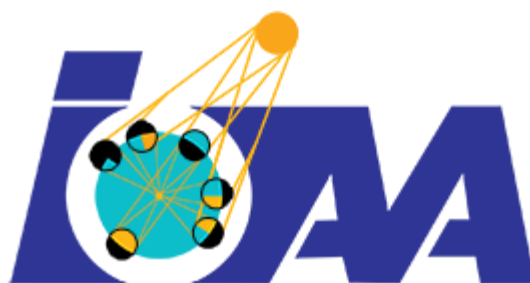


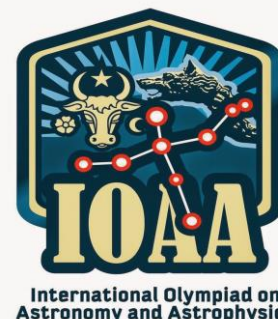
12th International Olympiad on
Astronomy and Astrophysics

IOAA Bisook

Questions and Solutions from IOAA 2018-2013 devidamente expropriados*



International Olympiad on
Astronomy and Astrophysics



Instructions

1. The theoretical competition will be 5 hours in duration and is marked out of a total of 300 points.
2. There are **Answer Sheets** for carrying out detailed work/rough work. On each **Answer Sheet**, please fill in
 - Student Code
 - Question No.
 - Page no. and total number of pages.
3. Start each problem on a separate Answer Sheet. Please write only on the printed side of the sheet. Do not use the reverse side. If you have written something on any sheet which you do not want to be marked, cross it out.
4. Please remember that the graders may not understand your language. As far as possible, write your solutions only using mathematical expressions and numbers. If it is necessary to explain something in words, please use short phrases (if possible in English).
5. You are not allowed to leave your working desk without permission. If you need any assistance (malfunctioning calculator, need to visit a restroom, need more Answer Sheets, etc.), please put up your hand to signal the proctor.
6. The beginning and end of the competition will be indicated by a long sound signal. Additionally, there will be a short sound signal fifteen minutes before the end of the competition (before the final long sound signal).
7. At the end of the competition you must stop writing immediately. Sort and put your sheets in separate stacks,
 - (a) Stack 1: answer sheets of part 1
 - (b) Stack 2: answer sheets of part 2
 - (c) Stack 3: answer sheets of part 3
 - (d) Stack 4: question papers and paper sheets that you do not want to be graded.
8. Wait at your table until your envelope is collected. Once all envelopes are collected, your student guide will escort you out of the competition room.
9. A list of constants for this competition is given on the next page.

Point distribution of this exam

Problem number	Points
T1	10
T2	10
T3	10
T4	10
T5	10
T6	25
T7	25
T8	25
T9	25
T10	75
T11	75
Total	300

Table of constants

Mass ($M_{\oplus}$)	5.98×10^{24} kg	Earth
Radius ($R_{\oplus}$)	6.38×10^6 m	
Acceleration of gravity (g)	9.81 ms^{-2}	
Obliquity of Ecliptic	$23^{\circ}27'$	
Length of Tropical Year	365.2422 mean solar days	
Length of Sidereal Year	365.2564 mean solar days	
Albedo	0.39	
Mass (M_{C})	7.35×10^{22} kg	Moon
Radius (R_{C})	1.74×10^6 m	
Mean Earth-Moon distance	3.84×10^8 m	
Orbital inclination with the Ecliptic	5.14°	
Albedo	0.14	
Apparent magnitude (mean full moon)	-12.74	
Mass ($M_{\odot}$)	1.99×10^{30} kg	Sun
Radius ($R_{\odot}$)	6.96×10^8 m	
Luminosity ($L_{\odot}$)	3.83×10^{26} W	
Absolute Magnitude	4.80 mag	
Surface Temperature	5772 K	
Angular diameter at Earth	$30'$	
Orbital velocity in Galaxy	220 kms^{-1}	
Distance from Galactic center	8.5 kpc	
1 au	1.50×10^{11} m	Physical constants
1 pc	206265 au	
Gravitational constant (G)	$6.67 \times 10^{-11} \text{ Nm}^2\text{kg}^{-2}$	
Planck constant (h)	6.62×10^{-34} Js	
Boltzmann constant (k_{B})	$1.38 \times 10^{-23} \text{ JK}^{-1}$	
Stefan-Boltzmann constant (σ)	$5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$	
Hubble constant (H_0)	$67.8 \text{ kms}^{-1}\text{Mpc}^{-1}$	
Speed of light in vacuum (c)	$299792458 \text{ ms}^{-1}$	
Magnetic Permeability of free space (μ_0)	$4\pi \times 10^{-7} \text{ Hm}^{-1}$	
1 Jansky (Jy)	$10^{-26} \text{ W m}^{-2}\text{Hz}^{-1}$	

Rayleigh-Jeans law is given by $B_{\nu} = \frac{2k_{\text{B}}T}{c^2} \nu^2$, which is the power emitted per unit emitting area, per steradian, per unit frequency.

(T1) Super Luminal Galaxies

(10 points)

Read the statements given below and state if they are true or false:

- (a) For some galaxies the apparent recession speed exceeds the speed of light.
- (b) The velocity – Distance relation as given by Hubble cannot allow recession velocities to exceed the speed of light.
- (c) Hubble-Lemaitre's law (formerly known as Hubble's Law) does not violate special relativity.
- (d) If some galaxies would have an apparent recession speed exceeding the speed of light, then the photons from those galaxies can never reach us.
- (e) As the expansion of Universe is accelerating, photons emitted right now from galaxies which have apparent recession speed equal to the speed of light will never reach us.

(T2) Distance

(10 points)

An observer measured trigonometric parallaxes of stars in a star cluster. Due to random errors, the measured parallax values are distributed symmetrically around the expected value with standard deviation equal to 0.05 mas (milliarcsec). Assume there are no systematic errors and assume all stars in the said cluster have the same luminosity. It is known that the distance of this cluster from us is $R = 5$ kpc.

He gave the data table to 4 of his students (A, B, C and D) and they estimated the distance to the cluster in the following ways:

- A. Convert each parallax measurement into distance and then find the average distance (R_A)
- B. Take the average of all parallaxes first and then convert the average parallax into distance. (R_B)
- C. Convert each parallax measurement into distance and then take the median distance value. (R_C)
- D. Find the median value of the parallaxes and then convert the median value into distance. (R_D)

State if the following statements are true or false. **In case a given mathematical relation is false, give the correct relation.**

- (l) If the i^{th} star gave the smallest value of parallax and the j^{th} star gave the highest value of parallax, in all likelihood $R_i - R > R - R_j$
- (m) $R_A = R$ (i.e. there is a high chance that the distance estimated by A fairly matches the true distance)
- (n) $R_B = R$ (i.e. there is a high chance that the distance estimated by B fairly matches the true distance)
- (o) $R_C < R$ (i.e. there is a high chance that the distance estimated by C will be systematically lower than the true distance)
- (p) $R_D = R$ (i.e. there is a high chance that the distance estimated by D fairly matches the true distance)

(T3) Atmospheric Refraction

(10 points)

Consider sunrise at Beijing ($\phi = 40^\circ$) on the vernal equinox day.

- (a) Let us say r_l , r_d , r_r and r_u are distances from the centre of the undistorted disk of the Sun to the edge of the disk towards the directions left, down, right and up respectively. What will be the hierarchical relation ($<$, $=$, $>$) between the four radii just after the sunrise?
- (b) What is the correction in the time of rise of the top edge of the disk as compared to the case without atmosphere? You may assume that typically atmospheric refraction near the horizon is $35'$. Please only consider the apparent diurnal motion.

(T4) Height of a Hill

(10 points)

Two friends wanted to measure the height of the hill next to their village (latitude $\varphi=40^\circ$). One of the friends climbed to the top of the hill and she agreed to send a light signal to her friend in the village as soon as she sees the sunset. On March 21, when they did this experiment, the friend in village received the light signal 4.1 minutes after the sunset from the village. Estimate the height of the hill and horizon distance for the person at the hill top. Ignore atmospheric refraction.

(T5) Sidereal Time

(10 points)

It is very interesting to observe that on one particular calendar day each year, the mean sidereal time will twice be 00:00:00.

- What will be the approximate R.A. of the Sun when this event happens?
- Estimate the exact date in 2018 for this event.

You may assume that at the Royal Greenwich Observatory, the mean sidereal time (GMST₀) was 6.706h at 0h, 1st January, 2018 (JD2458119.5).

(T6) Observe the Sun with FAST

(25 points)

The Five-hundred-meter Aperture Spherical radio Telescope (FAST) is a single-dish radio telescope located in Guizhou Province, China. The physical diameter of the dish is 500 m, but during observations, the effective diameter of the collecting area is 300 m.

Consider observations of the thermal radio emission from the photosphere of the Sun at 3.0 GHz with this telescope and a receiver with bandwidth 0.3 GHz.

- Calculate the total energy ($E_\odot$) that the receiver will collect during 1 hour of observation.
- Estimate the energy needed to turn over one page of your answer sheet (E'). Hint: the typical surface density of paper is 80 gm^{-2} .
- Which one is larger?

(T7) Sunspot

(25 points)

Magnetic fields are important in the physics of stars and sunspots. As an approximation, we can model the photosphere of the Sun consisting of a plasma, which can be simply treated as a single component ideal gas, and a magnetic field ($\mathbf{B}$), which has an associated magnetic pressure $p_B = \frac{B^2}{2\mu_0}$. It behaves like any other physical pressure except that it is carried by the magnetic field rather than by the kinetic energy of particles.

Assume that the number density of particles in the photosphere is constant everywhere, but the magnetic field inside the sunspot ($B_{\text{in}}=0.1\text{T}$) is much stronger than outside ($B_{\text{out}}=5\times 10^{-3}\text{T}$). From the blackbody spectrum, the temperature inside the sunspot is $T_{\text{in}}\sim 4000\text{K}$, while the temperature outside is $T_{\text{out}}\sim 6000\text{K}$ (which is why the sunspot looks darker). For the sunspot to be stable, the inside must be in equilibrium with the outside.

- Estimate the number density of plasma particles in the solar photosphere.
- Compare your answer with an estimate of the number density of particles in the atmosphere at the surface of the Earth.

(T8) A Possible Dark Matter Deficient Galaxy

(25 points)

Earlier this year, a team of astronomers reported their discovery of a galaxy with much less dark matter than the galaxy evolution model predicted (van Dokkum et al. 2018, Nature). This galaxy, named NGC 1052-DF2, is located close to the elliptical galaxy NGC 1052 ($D=20\text{Mpc}$ from the Sun) in the sky. The shape of NGC 1052-DF2 resembles an ellipse with semi major axis (a) of $22.6''$ and $\frac{b}{a} = 0.85$. Half of the total light from the galaxy comes from within this ellipse and the mean surface brightness within the ellipse is about $24.7 \text{ mag arcsec}^{-2}$.

- Calculate the total apparent magnitude of this galaxy.
- The team suggested the galaxy is a companion of NGC 1052. Determine the total mass of stars in NGC 1052-DF2, assuming it has a mass to light ratio $\left(\frac{M/M_\odot}{L/L_\odot}\right)$ of 2.0.
- The team identified 10 globular clusters in NGC 1052-DF2 with a mean galactocentric distance of $78.4''$. They also measured the velocity dispersion of these clusters to be not more than 8.4 km/s . Estimate the dynamical mass of this galaxy. For simplicity, assume the mass distribution in the galaxies is uniform and is spherically symmetric.
- This discovery was challenged by other groups (Kroupa et al., Nature, 2018, Truijlo et al., MNRAS, 2018), who claimed that NGC 1052-DF2 is not a satellite of NGC 1052, and it is located at a much smaller distance to us. Show why a smaller distance would weaken the assertion of the dark matter deficiency in NGC 1052-DF2.

(T9) Radio Galaxy

(25 points)

An observer wants to use the Five-hundred-meter Aperture Spherical radio Telescope (FAST) in China to observe a radio galaxy at redshift of $z = 0.06$. We assume that the radio source is compact compared to the beam size of the telescope at the observing frequencies, i.e., the source is point-like as seen through the telescope. To detect a point source with FAST, it must be sufficiently strong (bright) relative to the noise level (for single polarization observations), σ , which depends on the bandwidth, $\Delta\nu$, and the integration time (the radio astronomy equivalent of exposure time), t_i , as follows:

$$\sigma = \frac{2k_B T_{\text{sys}}}{A_e \sqrt{t_i \Delta\nu}}$$

where T_{sys} is the system temperature (about 150 K in the frequency range of $0.28 \text{ GHz} - 0.56 \text{ GHz}$ and 25 K in the frequency range of $1.05 \text{ GHz} - 1.45 \text{ GHz}$), and $A_e = 4.6 \times 10^4 \text{ m}^2$ is the effective area of the telescope taking into account the total efficiency of the instrument.

This radio galaxy has an observed continuum flux density of $f_\nu = 2.5 \times 10^{-3} \text{ Jy}$ at an observing frequency of 0.4 GHz . The bandwidth $\Delta\nu$ for the continuum observation centered at 0.4 GHz is $2.8 \times 10^8 \text{ Hz}$.

- In order to detect the continuum flux density at 0.4 GHz with a signal-to-noise ratio of 30 (a so-called 30σ detection), what is the required integration time, t_i ?
- We want to search for the neutral Hydrogen (HI) in the galaxy using 21cm absorption line. The HI 21cm line, with rest frame frequency of 1.4204 GHz . Calculate the observed frequency (ν_{obs}) of the HI line for this galaxy.
- The radio continuum emission from this galaxy can be described by a power law $f_\nu \sim \nu^\alpha$, with a spectral index of $\alpha = -0.2$. Calculate the continuum flux density at ν_{obs} for this galaxy.
- The line width of the HI 21cm absorption line is 90 km/s . Calculate the line width in Hz at the observing frequency of ν_{obs} . According to Figure 1, the HI 21cm line absorbs 4% of the continuum flux density (on average) over the line width of 90 km/s . In order to detect the absorption line at $\geq 3\sigma$ in three consecutive 30 km/s channels, what is the required integration time?

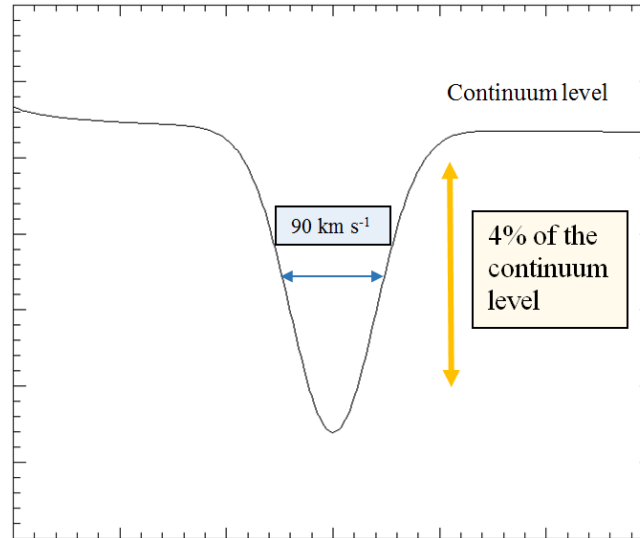


Figure 1: Spectrum of the HI 21cm absorption relative to the continuum emission in the radio galaxy

(T10) Vega and Altair

(75 points)

As per a very famous Chinese folklore about love, Vega and Altair are two lovers. It is said that they can meet each other once every year on a bridge made up of birds over the Milky Way. The parameters of two stars are given in the table below. For the purpose of this question, assume that the coordinate frame is fixed (i.e. not affected by precession or motion of the Sun).

Star	Right Ascension (J2000.0)	Declination (J2000.0)	Parallax (mas)	Proper Motion		Radial Velocity (km/s)
				$\mu_\alpha \cos \delta$ (mas/year)	μ_δ (mas/year)	
Vega	18 ^h 36 ^m 56.49 ^s	+38° 47' 07.7"	130.23	+200.94	+286.23	-13.9
Altair	19 ^h 50 ^m 47.70 ^s	+8° 52' 13.3"	194.95	+536.23	+385.29	-26.1

Based on this data, answer the following questions:

- (9 points) What is the angular separation of the two stars?
- (6 points) Calculate the distance (in parsecs) between Vega and Altair.
- (3 points) Calculate position angles of the proper motion vectors of each of these two stars.

For parts d-g, assume that the angular velocity of the stars on the celestial sphere remains constant. This is not a physical situation but this is an assumption to simplify the problem.

- (2 points) How many common points on the celestial sphere are there which can be reached by both these stars?
- (20 points) Find the coordinates of the closest such point.
(Note: Drawing the situation on a celestial sphere will help you in visualising the situation)
- (8 points) Find when (which year) each of these stars were / will be at that point.
- (5 points) When Altair was / will be at that point, what would be its angular separation from Vega?
- (22 points) Find coordinates of any point (if it exists) in 3-D space which was /will be visited by both these stars. Do not ignore radial velocities for this part of the question.

(T11) Thermal History of the Universe

(75 points)

Based on Einstein's general relativity, Russian physicist Alexander Friedmann derived the Friedmann Equation by which the dynamics of a homogeneous and isotropic universe can be well described. The Friedmann Equation is usually written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}.$$

We define the Hubble parameter as $= \frac{\dot{a}}{a}$, where a is the scale factor and $\dot{a}$ is the rate of change of scale factor with time. Thus, the Hubble parameter is a function of cosmic time. In the Friedmann Equation, ρ_m is the density of matter, including dark matter and baryons, ρ_r is the density of radiation, Λ is the cosmological constant, and k is the curvature of space. Subscript 0 indicates the value of a physical quantity at present day, e.g. H_0 is the present value Hubble parameter. Also, to avoid confusion with the reduced Hubble parameter, we use the reduced Planck Constant $\hbar = h/(2\pi)$ instead of the Planck constant h .

(a) (5 points) What are the dimensions of Hubble parameter? One can define a characteristic timescale for the expansion of the Universe (i.e. Hubble time t_H) using the Hubble parameter. Calculate the present-day Hubble time t_{H0} .

(b) (5 points) Let us define the critical density ρ_c as the matter density required to explain the expansion of a flat universe without any radiation or dark energy. Find an expression of the critical density, in terms H and G . Calculate the present critical density ρ_{c0} .

(c) (6 points) It is convenient to define all density parameters in a dimensionless manner like $\Omega_i = \frac{\rho_i}{\rho_c}$, i.e. the ratio of density to critical density. The Friedmann Equation can be rewritten using these dimensionless density parameters simply as, $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$.

Use this information to find expression for Ω_Λ and Ω_k , in terms H , c , Λ , k and a .

(d) (7 points) Another equation which is valid for matter, radiation and dark energy is often called the Fluid Equation: $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$, where p is the pressure of some component, ρ is the density and $\dot{\rho}$ is the rate of change of density over time. Radiation contains photons and massless neutrinos, and they both travel at the speed of light. The pressure exerted by these particles is 1/3 of their energy density. Show that the density of radiation $\rho_r \propto (1+z)^4$, where z is cosmological redshift. You may note that if $\frac{\dot{\rho}}{\rho} = n \frac{\dot{a}}{a}$, then $\rho \propto a^n$.

(e) (4 points) We know that the value of the cosmological constant Λ doesn't evolve. Its equation of state has a form $p = w\rho_\Lambda c^2$, where w is an integer. Find the value of w .

(f) (13 points) Planck time, defines a characteristic timescale before which our present physical laws are no longer valid, and where quantum gravity is needed. The expression for Planck time can be written in terms of $\hbar$, G and c and non-dimensional coefficient of this expression in SI units is of the order of unity. Using dimensional analysis, find expression for Planck time and estimate its value.

(g) (7 points) Planck length defines the length scale associated with Planck time is given by $l_p = ct_p$. The minimal mass of a black hole, also called Planck mass, is defined as the mass of a black hole whose Schwarzschild radius is two times the Planck length.

Derive the Planck mass M_p and calculate $M_p c^2$ in GeV. This mass is considered to be an upper threshold for elementary particles, beyond which they will collapse to a black hole.

(h) (4 points) At the very beginning (soon after the Planck time), all the particles were in thermal equilibrium in a primordial soup. As temperature decreased, different particles then decoupled from the primordial soup one by one and could travel freely in the Universe. Photons decoupled at ~ 300000 years after the Big Bang. These photons emitted at that time are what constitutes the cosmic microwave background (CMB), which follows the Stefan-Boltzmann law for blackbody radiation.

$$\epsilon_r = \frac{\pi^2}{15\hbar^3 c^3} (k_B T)^4,$$

Show that the temperature of the CMB follows $T/(1+z) = \text{constant}$.

(i) (16 points) With the expansion of the Universe, radiation density dropped more quickly than matter density, and at some epoch the matter density was equal to the radiation density. Radiation contains both photons and neutrinos. Apart from photons, neutrinos additionally contribute to the radiation energy density by 68% (i.e. $\Omega_{r0} = 1.68\Omega_{\gamma0}$, where γ indicates photons). Estimate the redshift of matter-radiation equality z_{eq} in terms of Ω_{m0} and reduced Hubble parameter $h = \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}}$. You may use the current temperature of the CMB: $T_0 = 2.73 \text{ K}$.

(j) (8 points) The neutrinos decoupled from the primordial soup when the temperature of the universe was around 1 MeV. At this time, the radiation density in the universe was much more than all other components. Estimate the time ($t = \frac{1}{2H}$) when neutrinos decoupled, and express it in seconds since the big bang.

Parte 1

(T1) A Grande Nuvem de Magalhães em Phuket

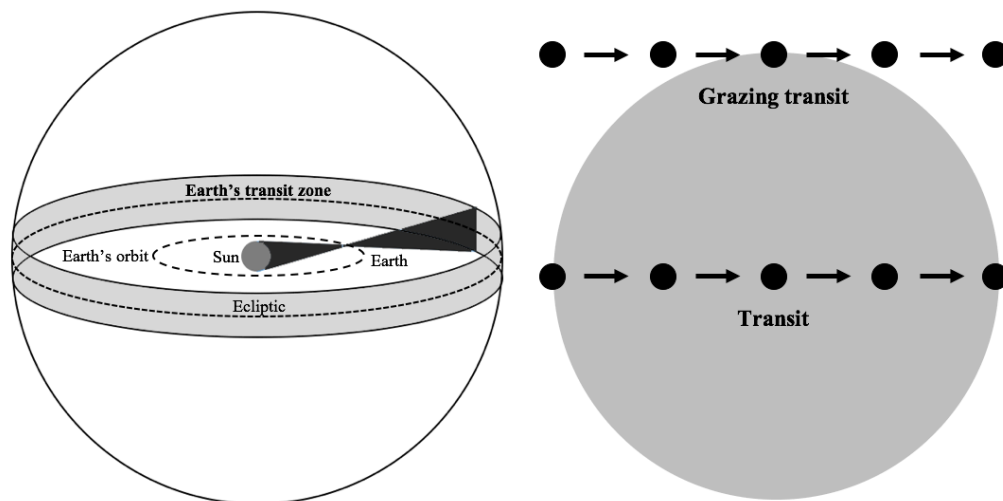
[10 pontos]

As coordenadas da Grande Nuvem de Magalhães (LMC) são R.A. = 5h 24min e Dec = $-70^{\circ} 00'$. A latitude e longitude de Phuket são $7^{\circ}53' N$ e $98^{\circ}24' E$, respectivamente. Em que data a LMC culmina às 21h em um determinado ano em Phuket? O Tempo Sideral em Greenwich (GST) às 00h de 1º de Janeiro é cerca de 6h 43min. Phuket está na zona horária UT+7. [10]

(T2) A região de trânsito da Terra

[10 pontos]

A região de trânsito da Terra é a área onde um observador extra-solar (situado bem longe do Sistema Solar) pode detectar o trânsito da Terra pelo disco solar. Para observadores na Terra, esta área é a projeção no plano do céu de uma faixa o redor da eclíptica (em cinza claro na figura da esquerda). Considere que a Terra possui órbita circular de raio 1 U.A.



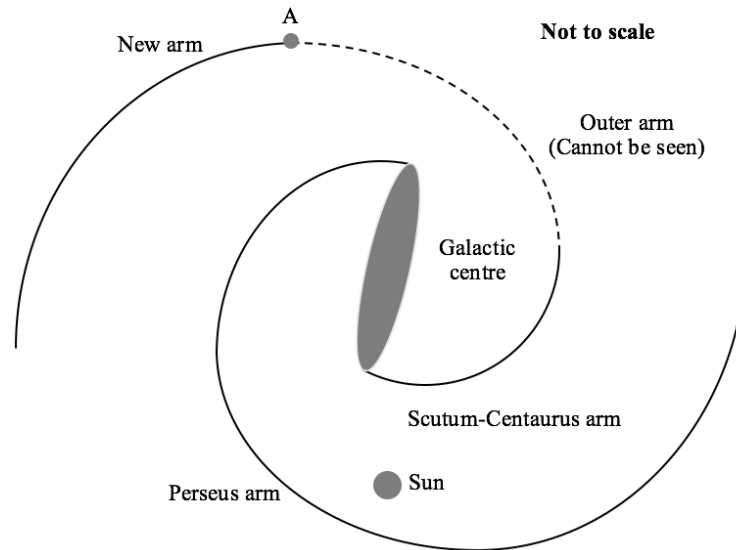
- Encontre a largura angular (em graus) da região de trânsito da Terra onde os observadores extra-solares podem detectar um trânsito total da Terra (i.e. todo o disco da Terra passa em frente ao Sol). [5]
- Encontre a largura angular (em graus) da região de trânsito da Terra onde os observadores extra-solares podem detectar pelo menos o trânsito parcial da Terra (i.e. apenas uma parte do disco da Terra passa em frente ao Sol). [5]

(T3) O Novo Braço Exterior da Via Láctea

[10 pontos]

Em 2011, Dame e Thaddeus descobriram uma nova parte do braço exterior da Via Láctea, estudando a linha de emissão de CO com o telescópio de 1,2 m do CfA. Eles observaram que a distribuição de CO se inicia longitude galáctica $\ell = 13,25^{\circ}$ (letra A na figura), onde sua velocidade radial é de $20,9 \text{ km.s}^{-1}$ se aproximando do Sol. Considere que a curva de rotação galáctica é plana após 5 kpc do Centro Galático. A distância entre o Sol e o Centro Galático é de 8,5 kpc. A velocidade do Sol ao redor do Centro Galático é de 220 km.s^{-1} .

- Encontre a distância do início do braço (ponto A) ao Centro Galático. [7]
- Encontre a distância do início do braço (ponto A) ao Sol. [3]



(T4) Levantamento de galáxias HI em 21 cm [10 pontos]

Um rádio telescópio está equipado com um receptor que pode observar na faixa de frequências de 1,32 a 1,52 GHz. O limite de detecção é de 0,5 mJy por feixe para um tempo de integração de 1 minuto. Em um levantamento de galáxias, a luminosidade da linha espectral de HI de uma galáxia típica é de 10^{28} W, com largura de linha de 1 MHz. Para um feixe tão grande, a região emissora de HI de uma galáxia distante pode ser aproximada por uma fonte pontual. A linha espectral da transição spin-flip do HI possui frequência no referencial de repouso de 1,42 GHz.

Qual o maior redshift z de uma galáxia HI típica que pode ser detectado por um levantamento realizado com este rádio telescópio com um tempo de integração de 1 minuto? Considere em seus cálculos que o redshift é pequeno e que portanto pode ser usada a aproximação não-relativística. Use $1 \text{ Jy} = 10^{-26} \text{ W.m}^{-2}.\text{Hz}^{-1}$. [10]

(T5) Satélite Geoestacionário [10 pontos]

Um satélite geoestacionário é um satélite que orbita a Terra com período orbital igual ao período de rotação terrestre. A altitude desses satélites é de 35786 km acima da superfície terrestre. Um satélite é colocado em uma órbita geoestacionária inclinada, com inclinação orbital de $\theta = 6,69^\circ$ com relação ao plano equatorial. Calcule o valor preciso da máxima altitude possível que o satélite atinge para um observador situado na latitude $\phi = 51,49^\circ$. Ignore a refração atmosférica. [10]

Parte 2

(T6) Supernova 1987A

[15 pontos]

A Supernova SN 1987A atingiu seu maior brilho com magnitude aparente de +3 em 15 de maio de 1987, e em seguida foi ficando mais fraca até ficar invisível a olho nu em 4 de fevereiro de 1988. Considere que seu brilho B variou com o tempo t como em um decaimento exponencial da forma $B = B_0 e^{-t/\tau}$, onde B_0 e τ são constantes. A magnitude aparente limite do olho humano nu é +6.

- Determine o valor de τ em dias. [5]
- Encontre o último dia em que um observador ainda poderia ver a supernova se estivesse utilizando um telescópio de 6" (15,24 cm) com eficiência de transmissão $T = 70\%$. Considere que o diâmetro médio da pupila humana é de 0,6 cm. [10]

(T7) Vida em outros planetas

[20 pontos]

Um dos lugares para procura de vida é em planetas orbitando estrelas de Sequência Principal. Um bom ponto de partida são os planetas que possuem faixa de temperatura semelhante à da Terra, com pequenas flutuações. Considere que, para uma estrela de Sequência Principal, a relação entre a luminosidade L e a massa M é dada por

$$L \propto M^{3,5}.$$

Considere que a energia total E liberada ao longo da vida de uma estrela é proporcional à massa M da estrela. O Sol ficará na Sequência Principal por cerca de 10 bilhões de anos. A tabela abaixo apresenta as massas de diferentes classes espectrais. Considere que as subclasses espectrais das estrelas (0-9) correlacionam-se linearmente com $\log M$.

Classe Espectral	O5V	B0V	A0V	F0V	G0V	K0V	M0V
Massa ($M / M_{\odot}$)	60	17,5	2,9	1,6	1,05	0,79	0,51

- Se considerarmos que são necessários pelo menos 4×10^9 anos para que uma forma de vida inteligente evolua, qual o tipo espectral (preciso até o nível da subclasse) da estrela mais massiva em torno da qual os astrônomos devem procurar por vida inteligente? [6]
- Considere que o planeta alvo tenha a mesma emissividade ϵ e albedo a da Terra. Escreva a expressão para a distância d do planeta, em U.A., até sua estrela da Sequência Principal, de massa M , para que ele tenha a mesma temperatura da Terra. [6]
- A existência de um planeta ao redor de uma estrela pode ser inferida pela variação na velocidade radial da estrela em relação ao centro de massa do sistema. Se o menor deslocamento Doppler no comprimento de onda detectável pelo observador é $(\Delta\lambda / \lambda) = 10^{-10}$, calcule a menor massa de um planeta como o do item (b), em unidades de massas terrestres, que pode ser detectado por este método, ao redor de uma estrela de Sequência Principal como a do item (a). [8]

(T8) Estrela de Belém

[20 pontos]

Uma grande conjunção é uma conjunção de Júpiter e Saturno para observadores na Terra. Considere que Júpiter e Saturno têm orbitas circulares no plano da Eclíptica.

O tempo entre conjunções sucessivas pode variar ligeiramente quando vistas da Terra. No entanto, o intervalo de tempo médio das grandes conjunções é o mesmo que o de um observador no centro do sistema solar.

- a) Encontre o período médio (em anos) e ângulo heliocêntrico médio (em graus) entre duas grandes conjunções sucessivas. [6]
- b) A próxima grande conjunção será em 21 de dezembro de 2020, com elongação de $30,3^\circ$ a leste do Sol. Estime em que constelação ocorrerá a conjunção em 21 de dezembro de 2020. Use o nome latino da IAU, ou a abreviatura de três letras da IAU da constelação, ou seja, Ursa Major ou UMa. [2]

Em 1606, Johannes Kepler determinou que em alguns anos a grande conjunção pode ser vista três vezes no mesmo ano devido aos movimentos retrógrados dos planetas. Ele também determinou que um evento destes aconteceu no ano 7 a.C., e que poderia ter sido o evento comumente conhecido como "A Estrela de Belém". Para os cálculos abaixo, você pode ignorar a precessão do eixo da Terra.

- c) Estime em que constelação ocorreu a grande conjunção de 7 a.C. Use o nome latino da IAU, ou a abreviatura de três letras da IAU da constelação, ou seja, Ursa Major ou UMa. [8]
- d) Na segunda conjunção da série de três conjunções de 7 a.C., para um observador na Terra, estime em que constelação estava o Sol. Use o nome latino da IAU, ou a abreviatura de três letras da IAU da constelação, ou seja, Ursa Major ou UMa. [4]

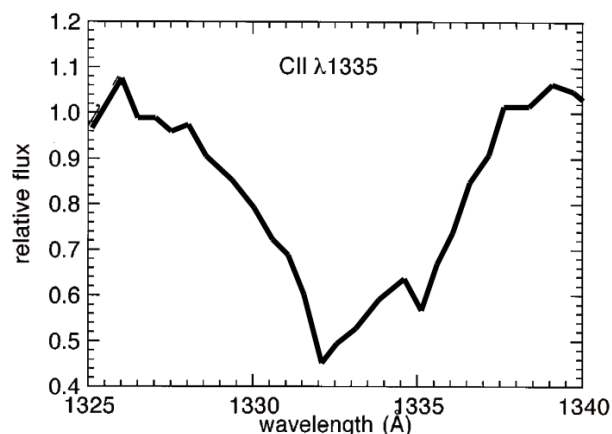
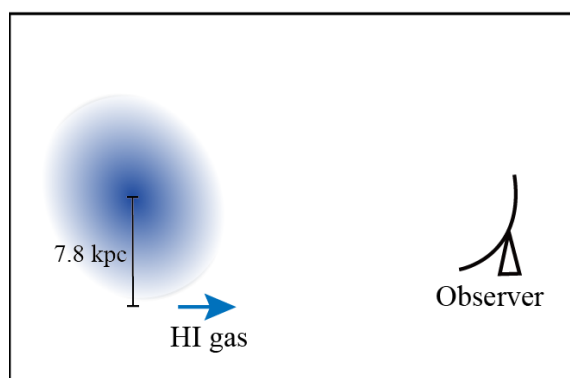
(T9) Fluxo Galático

[20 pontos]

Cannon et al. (2004) observaram HI no disco da galáxia starburst IRAS 0833+6517 com o Very Large Array (VLA). A galáxia está localizada a uma distância de 80,2 Mpc com um ângulo de inclinação aproximado de 23 graus. De acordo com o mapa de velocidades de HI, IRAS 0833+6517 parece apresentar uma rotação regular, com velocidade radial observada do gás HI de aproximadamente 5850 km.s^{-1} a uma distância de 7,8 kpc do centro (painel esquerdo da figura).

O fluxo de gás de IRAS 0833+6517 é mapeado usando linhas de absorção interestelar desviadas para o azul, observadas sobre o contínuo estelar (painel direito da figura). Supondo que esta galáxia seja gravitacionalmente estável, e que todas as estrelas se movam em órbitas circulares,

- Determine a velocidade de rotação (v_{rot}) de IRAS 0833+6517 no raio onde foi observado o gás HI. [5]
- Calcule a velocidade de escape da galáxia para uma partícula de teste do fluxo de gás no raio de 7,8 kpc. [9]
- Examine se o gás do fluxo pode escapar da galáxia nesse raio, levando em conta o deslocamento de velocidade da linha de absorção C II $\lambda 1335$. Esta linha já está corrigida para a velocidade de afastamento da galáxia. [6]



(T10) GOTO

[25 pontos]

O objetivo do Observador de Transientes Ópticos de Ondas Gravitacionais (GOTO, em inglês) é executar buscas por contrapartidas ópticas de fontes de Ondas Gravitacionais (GW) em até uma hora após a detecção pelos experimentos LIGO e VIRGO. A busca precisa cobrir uma grande área do céu em um tempo curto para vasculhar todas as possíveis regiões cobertas pelos experimentos de GW, antes que o sinal óptico da fonte desapareça. A rede de telescópios GOTO é composta por 4 telescópios refletores idênticos, cada um com 40 cm de abertura e razão focal $f/2,5$, que trabalham em conjunto para imagear regiões amplas do céu. Para simplificar, considere que os campos de visão (FoV) dos telescópios não se sobrepõem.

- Calcule o tamanho angular projetado por mm no plano focal de cada telescópio, i.e. a escala de placa. [6]
- Se a magnitude de ponto zero, (i.e. a magnitude para qual a taxa de contagens do detector é de 1 contagem por segundo) do sistema de telescópios é de 18,5 mag, calcule o tempo mínimo para que se atinja magnitude 21 mag para uma fonte pontual com Razão Sinal-Ruído (SNR) = 5. Considere que o ruído é dominado pelo Ruído de Leitura (RON) em 10 contagens/pixel, e pelo ruído térmico do CCD (DN) a uma taxa de 1 contagem/pixel/minuto. Os CCDs utilizados no GOTO possuem pixels de 6 microns de lado e ganho (fator de conversão entre os fotoelétrons e as contagens) de 1. O seeing típico no observatório é 1,0 segundo de arco. [8]

A Razão Sinal-Ruído é definida por

$$\text{SNR} \equiv \frac{\text{Total de Contagens da Fonte}}{\sqrt{\sum_i \text{Ruído}_i^2}} = \frac{\text{Total de Contagens da Fonte}}{\sqrt{\sigma_{\text{RON}}^2 + \sigma_{\text{DN}}^2 + \dots}},$$
$$\sigma_{\text{RON}} = \sqrt{N_{\text{pix}} \cdot \text{RON}^2}, \quad \sigma_{\text{DN}} = \sqrt{N_{\text{pix}} \cdot \text{DN} \cdot t},$$

onde t é o tempo de exposição.

- c) Normalmente, quando o tempo de exposição é longo e as contagens da fonte são altas, o Ruído de Poisson da fonte também é significativo. Determine a relação entre SNR e tempo de exposição no caso em que o ruído é dominado pelo Ruído de Poisson da fonte. Calcule novamente o tempo de exposição mínimo para atingir magnitude 21 mag com SNR=5, como foi feito no item anterior, se também considerarmos o Ruído de Poisson.

O Ruído de Poisson da fonte é dado por $\sigma_{\text{fonte}} = \sqrt{\text{Contagens da Fonte}}$.

Obs: na prática, o fundo do céu também pode ser uma fonte importante de Ruído de Poisson, mas neste exercício pode ser desprezado. [6]

- d) A incerteza de localização típica de um detector de GW é de cerca de 100 graus quadrados. Gostaríamos de cobrir toda essa área em no máximo 1 hora após a detecção da GW. Estime o menor comprimento possível (em pixels) do lado de um CCD quadrado, a ser instalado em cada um dos telescópios, para que isso aconteça. Considere que os tempos de leitura dos CCDs e de apontamento dos telescópios entre um campo e outro são desprezíveis. [5]

Parte 3

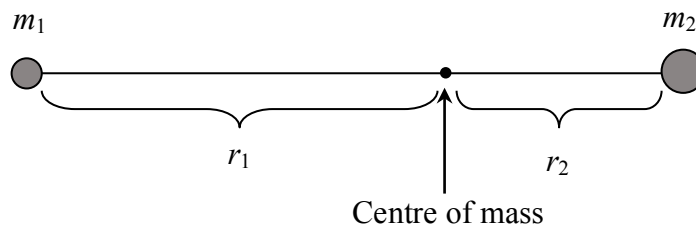
(T11) Massa do Grupo Local de Galáxias

[50 pontos]

A dinâmica de Galáxia de Andrômeda (M31) e da Via Láctea (MW) pode ser usada para estimar a massa total do Grupo Local (LG). A ideia básica é que as galáxias, atualmente em um sistema binário, estavam aproximadamente no mesmo ponto no espaço logo após o Big Bang. Além disso, a massa do Grupo Local é dominada pelas massas da MW e M31. Através do deslocamento Doppler das linhas espectrais, verificou-se que M31 está se movendo em direção à MW com uma velocidade de 118 km.s^{-1} . Isso pode ser surpreendente, dado que a maioria das galáxias se afastam uma das outras, de acordo com a Lei de Hubble. O fato de que o M31 está se movendo em direção à MW deve-se, presumidamente, à atração gravitacional mútua que, eventualmente, inverteu suas velocidades iniciais. Em princípio, se o par de galáxias for bem representado por massas pontuais isoladas, sua massa total pode ser determinada medindo sua separação, velocidade relativa e o tempo desde o início do Universo. Kahn e Woltjer (1959) usaram esse argumento para estimar a massa do LG.

Neste problema, usaremos esse argumento ao longo de nossos cálculos da seguinte maneira.

- a) Considere um sistema isolado com momento angular desprezível, composto por duas massas pontuais m_1 e m_2 (como observado por um observador inercial no centro de massa do sistema).



Escreva a expressão matemática para a Energia Mecânica Total (E) desse sistema, relacionando m_1 , m_2 , r_1 , r_2 , v_1 , v_2 , e a constante gravitacional G , onde v_1 e v_2 são as velocidades radiais de m_1 e m_2 , respectivamente. [5]

- b) Reescreva a equação do item (a) em termos de r , v , μ , M , e G , onde $r \equiv r_1 + r_2$ é a distância de separação entre m_1 e m_2 , v é a taxa de variação dessa distância, $\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$ é a massa reduzida do sistema, e $M \equiv m_1 + m_2$ é a massa total do sistema. [10]

- c) Mostre que a equação do item (b) resulta em $v^2 = (2GM) \left(\frac{1}{r} - \frac{1}{r_0} \right)$, onde r_0 é uma nova constante.

Encontre r_0 em termos de μ , M , G e E . [5]

A solução da equação do item (b) é dada abaixo numa forma paramétrica, sob a condição inicial $r = 0$ em $t = 0$:

$$r(\theta) = \frac{r_0}{2} (1 - \cos \theta),$$
$$t(\theta) = \left(\frac{r_0^3}{8GM} \right)^{\frac{1}{2}} (\theta - \sin \theta),$$

onde θ está em radianos.

- d) A partir das equações paramétricas acima, mostre que uma expressão para $\frac{vt}{r}$ é

$$\frac{vt}{r} = \frac{(\sin \theta)(\theta - \sin \theta)}{(1 - \cos \theta)^2}. \quad [10]$$

- e) Considere agora que m_1 e m_2 são as massas da MW e M31 respectivamente. Os valores atuais de v e r são $v = -118 \text{ km s}^{-1}$ e $r = 710 \text{ kpc}$, e t pode ser considerado como a idade do Universo (13700 milhões de anos). Encontre θ por iteração numérica. [10]

- f) Use o valor de θ obtido no item (e) para obter o valor de $r_{\max}$. Obtenha também o valor de M em massas solares. [10]

(T12) Naufrágio

[40 pontos]

Você é um naufrago em uma ilha. Felizmente, você ainda está usando um relógio, que está configurado para a hora de Bangkok. Você também tem uma bússola, um atlas e uma calculadora. Inicialmente você está inconsciente, e ao acordar descobre que anoiteceu há pouco tempo. Infelizmente o tempo está nublado. Depois de pouco mais de uma hora, você vê Orion através de um buraco nas nuvens. Você estima que a estrela "Rigel" está a cerca de $52,5^\circ$ acima do horizonte e, com a sua bússola, descobre que ela tem azimute astronômico de 109° . Seu relógio marca 01:00 de 21 de novembro de 2017. Você, então, se lembra das suas aulas de astronomia que o Tempo Sideral de Greenwich (GST) às 00h de 1º de janeiro de 2017 é cerca de 6h 43 min, e que ascensão reta e declinação de Rigel são 5h 15min e $-8^\circ 11'$, respectivamente. Bangkok está na zona horária UT +7.

- a) Encontre o Ângulo Horário Local (LHA) de Rigel. [10]
- b) Encontre o Tempo Sideral de Greenwich (GST) atual. [10]
- c) Encontre a longitude da ilha. [5]
- d) Encontre a latitude da ilha, com precisão de um minuto de arco. [15]

(T13) Exolua

[60 pontos]

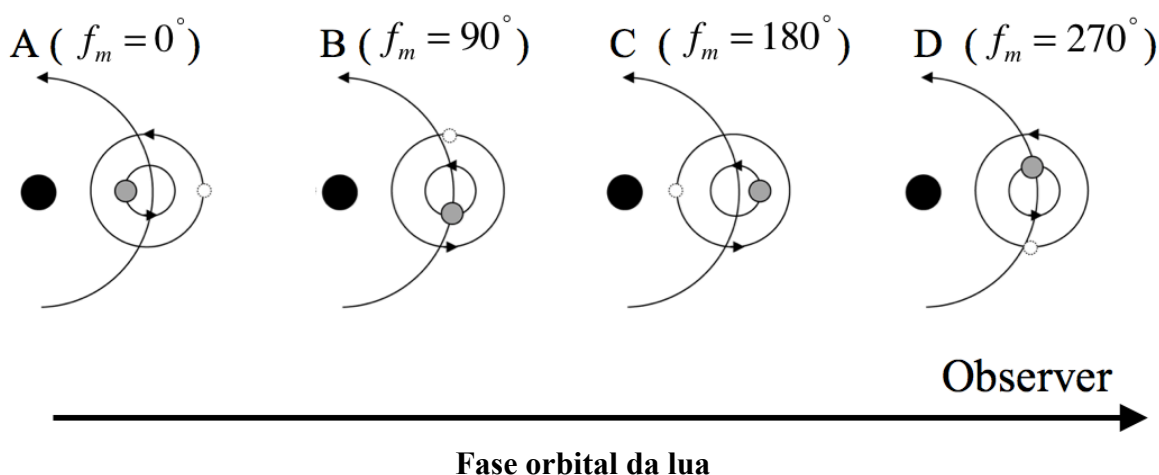
Exoluas são satélites naturais de exoplanetas. A influência gravitacional de tal lua afetará a posição do planeta em relação ao baricentro planeta-lua, resultando em Variações no Tempo de Trânsito (σ_{TTV} , TTVs), pois o trânsito observado do planeta acontece mais cedo ou mais tarde do que o trânsito previsto para um planeta sem lua.

O movimento do planeta ao redor do baricentro planeta-lua também induzirá Variações no Duração de Trânsito (σ_{TDV} , TDVs), uma vez que a duração do trânsito observada é mais curta ou mais longa que a prevista para um planeta sem lua.

Vamos considerar uma órbita circular, vista de perfil, com os seguintes parâmetros:

- M_p é a massa do planeta
- M_m é a massa da lua
- P_p é o período orbital do baricentro planeta-lua ao redor da estrela hospedeira
- P_m é o período orbital da lua ao redor do planeta
- a_p é a distância do baricentro planeta-lua à estrela
- a_m é a distância da lua ao baricentro planeta-lua
- f_m é a fase orbital da lua, com $f_m = 0^\circ$ quando a lua está em oposição à estrela
- τ é a duração média do trânsito do planeta (como se ele não tivesse lua)

Iremos considerar apenas a órbita prógrada da lua, no mesmo plano da órbita do planeta. Exemplos de fases orbitais da lua, como observadas por observadores distantes, são mostrados na figura abaixo.



Os círculos preto, cinza e branco representam a estrela, planeta e lua, respectivamente.

- a) Definimos $\sigma_{TTV} \equiv t_m - t$, onde t é o tempo previsto de trânsito sem a lua, e t_m é o tempo observado de trânsito com a lua. Mostre que

$$\sigma_{TTV} = \left[\frac{a_m M_m P_p}{2\pi a_p M_p} \right] \sin(f_m)$$

Valores positivos de σ_{TTV} indicam que o trânsito ocorre depois do tempo previsto do trânsito para um planeta sem lua. [10]

b) Seja $\sigma_{TDV} \equiv \tau_m - \tau$, onde τ é a duração prevista do trânsito sem a lua, e τ_m é a duração observada do trânsito com a lua. Podemos considerar que a velocidade do planeta ao redor da estrela é muito maior que a velocidade da lua ao redor do baricentro planeta-lua, e também que a lua não muda de fase durante o trânsito. Mostre que

$$\sigma_{TDV} = \tau \left[\frac{P_p M_m a_m}{P_m M_p a_p} \right] \cos(f_m)$$

Valores positivos de σ_{TDV} indicam que a duração do trânsito é mais longa que a prevista para um trânsito sem lua. [13]

O trânsito de um exoplaneta é observado numa estrela da Sequência Principal de tipo solar ($1M_{\odot}$, $1R_{\odot}$, Classe espectral: G2V). O planeta tem uma órbita circular vista de perfil com período de 3,50 dias. A partir dos dados observacionais, sabe-se que o planeta possui uma massa de $120 M_{\oplus}$ e um raio de $12 R_{\oplus}$. A relação observada entre σ_{TTV}^2 e σ_{TDV}^2 pode ser escrita como

$$\sigma_{TDV}^2 = -0,7432\sigma_{TTV}^2 + 1,933 \times 10^{-8} \text{ dias}^2$$

c) Considere que a massa da lua é muito menor que a do planeta. Encontre a duração média do trânsito do planeta (τ) em dias. [6]

d) Encontre o período da lua (P_m) em dias. [7]

e) Estime a distância da lua ao baricentro planeta-lua (a_m) em unidades de raios terrestres. Também encontre a massa da lua (M_m) em unidades de massas terrestres. [7]

f) A esfera de Hill é a região ao redor de um planeta dominada pela gravidade do planeta. O raio da esfera de Hill pode ser escrito como

$$R_h = a_p \sqrt[3]{\frac{M_p}{xM_*}}$$

Encontre o valor da constante x (Dica: para uma estrela massiva, o raio da esfera de Hill do sistema é aproximadamente a distância entre o planeta e o ponto de Lagrange L_1)

ou L_2). Expresse o raio da esfera de Hill desse sistema planetário em unidades de raios terrestres. [11]

- g) O limite de Roche é o raio orbital mínimo que um satélite pode orbitar sem ser destruído pelas forças de maré, e pode ser escrito como

$$R_r = 2,44 R_p \sqrt[3]{\frac{\rho_p}{\rho_m}}$$

onde ρ_p e ρ_m são as densidades do planeta e da lua, respectivamente. Considerando que a lua tem composição rochosa com densidade igual à da Terra, encontre o limite de Roche do sistema [3]

- h) A lua possui órbita estável? [3]

(T1) **Verdadeiro ou Falso**

Avalie se as seguintes afirmações são verdadeiras ou falsas. Marque suas respostas para cada item na folha de respostas (**TRUE / FALSE**). Não há necessidade de justificar suas respostas.

- (T1.1) Em uma fotografia de longa exposição do céu limpo numa noite de Lua Cheia, o céu aparecerá de cor azul, como durante o dia. 2
- (T1.2) Um astrônomo em Bhubaneswar registra a posição do Sol no céu, ao longo do ano, todos os dias às 05:00 UT. Se o eixo de rotação da Terra fosse perpendicular ao plano de sua órbita, estas posições descreveriam um arco de círculo máximo. 2
- (T1.3) Se o período orbital de um asteroide ao redor do Sol no plano da Eclíptica for menor do que o período de Urano, então sua órbita deve necessariamente estar toda contida no interior da órbita de Urano. 2
- (T1.4) O centro de massa do Sistema Solar está sempre dentro do Sol. 2
- (T1.5) Um fóton propaga-se livremente pelo espaço vazio. À medida que o Universo se expande, o *momentum* do fóton diminui. 2

(T2) **Gases em Titã**

10

As partículas de gás numa atmosfera planetária apresentam distribuições de velocidades bem amplas. Se a velocidade quadrática média (r.m.s.) térmica das partículas de um determinado gás for superior a 1/6 da velocidade de escape, então quase todo este gás irá escapar da atmosfera do planeta. Qual deve o peso atômico mínimo (em unidades de massa atômica) A_{min} de um gás monoatômico ideal para que ele permaneça na atmosfera de Titã?

Dados: massa de Titã $M_T = 1,23 \times 10^{23}$ kg, raio de Titã $R_T = 2575$ km, temperatura superficial de Titã $T_T = 93,7$ K.

(T3) **Universo Primordial**

Modelos cosmológicos indicam que a densidade de energia de radiação ρ_r no Universo é proporcional a $(1+z)^4$, e que a densidade de energia de matéria ρ_m é proporcional a $(1+z)^3$, onde z é o *redshift*. O parâmetro adimensional de densidade Ω é dado por $\Omega = \rho/\rho_c$, onde ρ_c é a densidade crítica de energia do Universo. No Universo atual, os parâmetros de densidade de radiação e de matéria são $\Omega_{r_0} = 10^{-4}$ e $\Omega_{m_0} = 0.3$, respectivamente.

- (T3.1) Calcule o *redshift* z_e no qual as densidades de energia da radiação e matéria são iguais. 3
- (T3.2) Supondo que a radiação do Universo primordial tem um espectro de corpo negro com temperatura 2.732 K, estime a temperatura T_e dessa radiação no *redshift* z_e . 4
- (T3.3) Estime a energia típica E_ν (em eV) dos fótons da radiação emitida no *redshift* z_e . 3

(T4) **Sombras**

10

Um observador no hemisfério norte reparou que o comprimento da menor sombra de uma vareta vertical de 1.000 m num determinado dia foi de 1.732 m. No mesmo dia, o comprimento da maior sombra foi de 5.671 m.

Encontre a latitude ϕ do observador e a declinação do Sol $\delta_\odot$ nesse dia. Considere o Sol como uma fonte puntiforme, e ignore a refração atmosférica.

(T5) **Trânsito no feixe do GMRT**

10

O *Giant Metrewave Radio Telescope* (GMRT), um dos maiores radiotelescópios do mundo em comprimentos de onda métricos, está localizado no Oeste da Índia (latitude: 19°6' N, longitude: 74°3' E). O GMRT possui 30 antenas parabólicas, cada uma com diâmetro de 45.0 m. Uma única antena do GMRT foi mantida fixa com seu eixo apontado para um ângulo zenital de 39°42' ao longo do meridiano norte, de modo que uma fonte de rádio puntiforme atravessasse o diâmetro de seu feixe no momento de sua passagem meridiana.

Qual a duração de trânsito T_{transit} no qual a fonte permanece dentro do feixe de largura a meia altura (FWHM- *full width at half maximum*) de uma única antena do GMRT? A observação é feita na frequência de 200 MHz.

Dica: O tamanho do feixe FWHM de uma antena de rádio operando a uma determinada frequência corresponde à resolução angular da antena. Considere que a iluminação é uniforme.

(T6) **Pulsações de Cefeidas**

A estrela β -Doradus é uma estrela variável Cefeida com um período de pulsação de 9.84 dias. Vamos fazer uma simplificação na qual a estrela brilha mais quando está mais contraída (raio R_1) e menos brilhante quando está mais expandida (raio R_2). Considere que a estrela mantém sua forma esférica, e que se comporta com um corpo negro durante todo seu ciclo. A magnitude bolométrica da estrela varia de 3.46 a 4.08. A partir de medidas Doppler, sabemos que durante a pulsação a superfície da estrela se expande e se contrai a uma velocidade radial média de 12.8 km s^{-1} . Ao longo do período de pulsação, o pico da radiação térmica intrínseca da estrela varia de 531.0 nm a 649.1 nm.

(T6.1) Encontre a razão dos raios da estrela nos seus estados de maior contração e expansão (R_1/R_2). 7

(T6.2) Encontre o raio da estrela (em metros) nos seus estados de maior contração e expansão (R_1 e R_2). 3

(T6.3) Calcule o fluxo F_2 da estrela no seu estado de maior expansão. 5

(T6.4) Encontre a distância D_{star} até a estrela, em parsecs. 5

(T7) **Óptica de Telescópios**

Num telescópio refrator ideal de razão focal $f/5$, a distância focal da objetiva é de 100 cm, e a da ocular 1 cm.

(T7.1) Qual é o aumento (ou ampliação angular) m_θ do telescópio? Qual é o comprimento focal do telescópio L_θ , ou seja, a distância entre a objetiva e a ocular? 4

A introdução de uma lente côncava (Barlow) entre a objetiva e o foco primário é uma técnica comum para se obter um maior aumento sem ter que se aumentar o comprimento focal do telescópio. Uma lente Barlow, de distância focal 1 cm, é então colocada no telescópio e dobra sua ampliação.

(T7.2) A que distância d_B do foco principal a lente Barlow deve ser colocada para se conseguir o dobro da ampliação? 6

(T7.3) Qual é o incremento ΔL no comprimento focal do telescópio? 4

Um telescópio é montado com a mesma lente objetiva do anterior e um CCD colocado no plano focal (sem a lente Barlow ou ocular). O tamanho de cada pixel do CCD é de $10 \mu\text{m}$.

(T7.4) Qual será a separação n_p , em pixels, entre os centróides de 2 estrelas no CCD, se elas estiverem separadas por $20''$ no céu? 6

(T8) **Fotometria na Banda U**

Uma estrela tem magnitude aparente $m_U = 15.0$ na banda U . O filtro da banda U é ideal, ou seja, tem transmissão perfeita (100%) dentro da banda, e é completamente opaco (0% de transmissão) fora dela. O filtro está centrado em 360 nm, e tem largura de banda de 80 nm. Considere que a estrela possui espectro de energia achatado com relação à frequência. A conversão entre a magnitude m em qualquer banda e a densidade de fluxo f de uma estrela em Jansky ($1 \text{ Jy} = 1 \times 10^{-26} \text{ W Hz}^{-1} \text{ m}^{-2}$) é dada por

$$f = 3631 \times 10^{-0.4m} \text{ Jy}$$

(T8.1) Aproximadamente quantos fótons N_0 na banda U provenientes dessa estrela irão incidir normalmente numa área de 1 m^2 no alto da atmosfera da Terra por segundo? 8

Essa estrela está sendo observada na banda U utilizando um telescópio em solo, cujo espelho primário tem 2.0 m de diâmetro. A extinção atmosférica na banda U durante a observação é de 50%. Considere

que o *seeing* é limitado pela difração. O brilho médio do céu noturno na banda U medido foi de 22.0 mag/arcsec².

(T8.2) Qual a razão R do número de fótons recebidos por segundo da estrela com relação aos fótons provenientes do céu, quando medidos sobre uma abertura circular de diâmetro $2''$? 8

(T8.3) Na prática, apenas 20% dos fótons na banda U que incidem no espelho primário são detectados. Quantos fótons N_t provenientes da estrela são detectados por segundo? 4

(T9) **Missão Mars Orbiter**

A sonda indiana Mars Orbiter Mission (MOM) foi lançada usando o Veículo Lançador de Satélites Polar (PSLV) em 5 de novembro de 2013. A carga útil da MOM (corpo + instrumentos) era de 500 kg, e a sonda carregava 852 kg de combustível. Ela foi inicialmente colocada numa órbita elíptica ao redor da Terra, com perigeu a 264.1 km de altitude e apogeu a 23903.6 km de altitude. Após aumentar sua órbita em 6 (seis) vezes, a MOM foi transferida para uma órbita de injeção trans-Marciana (órbita de Hohman).

O primeiro destes aumentos de órbita foi realizado disparando-se os motores por um curto período de tempo, perto do perigeu. Os motores foram disparados para mudar a órbita, mas sem mudar o plano da órbita, e sem alterar o perigeu. Isso deu um impulso de $1,73 \times 10^5 \text{ kg m s}^{-1}$ ao satélite. Despreze a mudança de massa durante a queima de combustível.

(T9.1) Qual será a altitude h_a do novo apogeu depois desse disparo do motor? 14

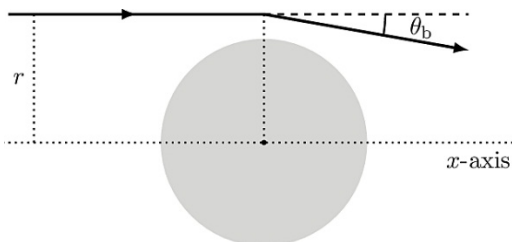
(T9.2) Encontre a excentricidade (e) da nova órbita e o novo período orbital (P) da MOM em horas, após realizado esse disparo. 6

(T10) **Telescópio de Lentes Gravitacionais**

A Teoria da Relatividade Geral de Einstein prevê o desvio da luz ao redor de corpos massivos. Para simplificar, consideremos o caso em que o desvio da luz ocorre apenas em um único ponto para cada raio luminoso, como mostrado na figura abaixo. O ângulo de desvio θ_b é dado por

$$\theta_b = \frac{2R_{\text{Sch}}}{r}$$

onde R_{Sch} é o raio de Schwarzschild associado ao corpo. Chamamos de “parâmetro de impacto” a distância r que separa o raio de luz incidente e o eixo x (paralelo ao raio) que passa pelo centro do corpo.



Dessa maneira, um corpo massivo se comporta como uma lente convergente. Os raios de luz, vindos do infinito, que têm mesmo parâmetro de impacto r convergem num ponto, ao longo do eixo x , a uma distância f_r do centro do corpo. Um observador neste ponto se beneficiará de uma imensa amplificação graças a esta focalização gravitacional. Neste caso, o corpo massivo é usado como um Telescópio de Lente Gravitacional para amplificação de sinais distantes.

(T10.1) Considere a possibilidade do nosso Sol ser um telescópio de lente gravitacional. Calcule a menor distância ao centro do Sol (em U.A.) f_{min} na qual os raios de luz podem ser focalizados. 6

(T10.2) Considere um pequeno detector circular de raio a , colocado a uma distância f_{min} sobre o eixo x e perpendicular a ele. Note que apenas os raios de luz que passam por um determinado *annulus* (anel) de largura h (onde $h \ll R_{\odot}$) ao redor do Sol irão chegar ao detector. O fator de amplificação no detector é definido como a razão da intensidade da luz incidente no detector na presença do Sol e a intensidade na ausência do Sol. 8

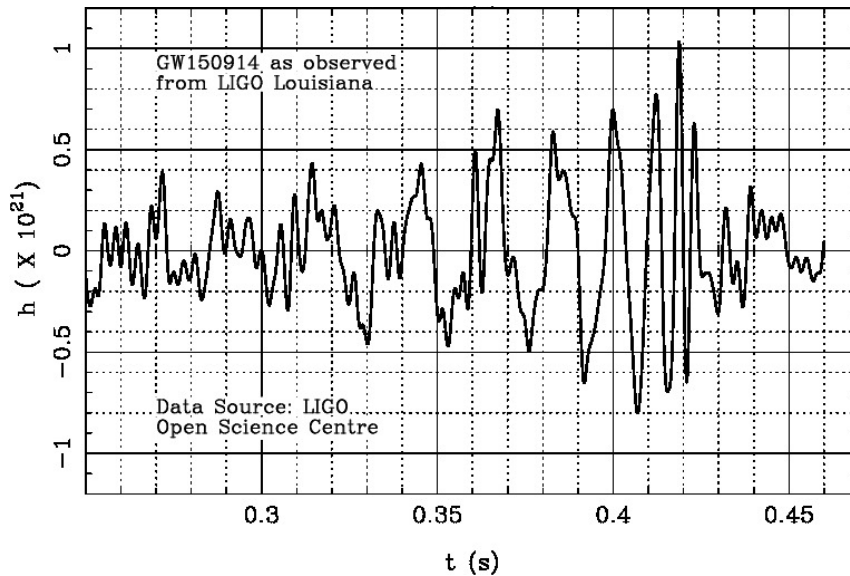
Expresse o fator de amplificação A_m no detector em termos de $R_\odot$ e α .

- (T10.3) Considere uma distribuição esférica de massa, tal como a de um aglomerado de matéria escura em um aglomerado de galáxias, através da qual os raios de luz podem passar enquanto sofrem desvio gravitacional. Considere, para simplificar, que para o parâmetro de impacto r apenas a massa $M(r)$ dentro do raio r é relevante.
Qual deve ser a distribuição de massa $M(r)$ para que a lente gravitacional se comporte como uma lente óptica convexa ideal?

6

(T11) Ondas Gravitacionais

Os primeiros sinais de ondas gravitacionais foram observados, em setembro de 2015, por dois detectores avançados do LIGO em Hanford e Livingston, EUA. Uma destas medidas (amplitude vs tempo, em segundos) é mostrada na figura a seguir. Nesta questão, vamos interpretar este sinal em termos de uma pequena massa de teste m orbitando uma grande massa M ($m \ll M$), considerando alguns modelos para a natureza da massa central.



A massa de teste perde energia devido à emissão de ondas gravitacionais. Como resultado, sua órbita vai decaindo, até que a massa atinja a superfície do objeto central, ou, no caso de um buraco negro, a órbita circular estável mais interna – ISCO, em inglês – que é dada por $R_{\text{ISCO}} = 3R_{\text{sch}}$, onde R_{sch} é o raio de Schwarzschild do buraco negro. Este é o “momento da coalescência”. Neste ponto, a amplitude da onda gravitacional é máxima, e também sua frequência, que é sempre o dobro da frequência orbital. Neste problema vamos considerar apenas as ondas gravitacionais antes da coalescência, quando as Leis de Kepler ainda são válidas. Após a coalescência a forma das ondas gravitacionais muda drasticamente.

- (T11.1) Considere as ondas gravitacionais mostradas na figura acima. Estime o período T_0 e a frequência f_0 das ondas gravitacionais nos instantes que precederam a coalescência.
- (T11.2) Para estrelas da Sequência Principal (MS), o raio estelar R_{MS} e a massa estelar M_{MS} estão relacionados através da seguinte lei de potência,

3

$$\begin{aligned} R_{\text{MS}} &\propto (M_{\text{MS}})^\alpha \\ \text{onde } \alpha &= 0.8 \quad \text{para } M_\odot < M_{\text{MS}} \\ &= 1.0 \quad \text{para } 0.08M_\odot \leq M_{\text{MS}} \leq M_\odot \end{aligned}$$

10

Se o objeto central for uma estrela da sequência principal, escreva a expressão para a frequência máxima da onda gravitacional f_{MS} , em função da massa estelar, em unidades de massa solar ($M_{\text{MS}}/M_\odot$), e de α .

- (T11.3) Utilizando o resultado anterior, determine o valor apropriado de α que dá a maior frequência possível para as ondas gravitacionais $f_{MS,max}$, para qualquer estrela da Sequência Principal. Calcule o valor desta frequência. 9
- (T11.4) Anãs brancas (WD) têm massa máxima de $1.44 M_{\odot}$ (conhecido como limite de Chandrasekhar) e obedecem a relação massa-raio $R \propto M^{-1/3}$. O raio de uma anã branca de 1 (uma) massa solar é de 6000 km. Encontre a maior frequência $f_{WD,max}$ das ondas gravitacionais emitidas, se a massa de teste estiver orbitando esta anã branca. 8
- (T11.5) Estrelas de nêutrons (NS) são um tipo peculiar de objeto compacto, cujas massas variam entre 1 e $3M_{\odot}$, com raios entre 10 a 15 km. Encontre a faixa de frequências $f_{NS,min}$ e $f_{NS,max}$ das ondas gravitacionais emitidas, se a massa de teste estiver orbitando uma estrela de nêutrons próxima ao raio dessa estrela. 8
- (T11.6) Se a massa de teste estiver orbitando um buraco negro (BH), escreva a expressão para a frequência f_{BH} da onda gravitacional emitida, em função da massa do buraco negro M_{BH} e da massa do Sol $M_{\odot}$. 7
- (T11.7) Baseado apenas no período (ou frequência) das ondas gravitacionais antes do momento da coalescência, determine se o objeto central pode ser uma estrela da Sequência Principal (MS), uma anã branca (WD), uma estrela de nêutrons (NS), ou um buraco negro (BH). Marque a opção correta na Folha de respostas. Estime a massa deste objeto M_{obj} em unidades de massa solar $M_{\odot}$. 5

(T12) Exoplanetas

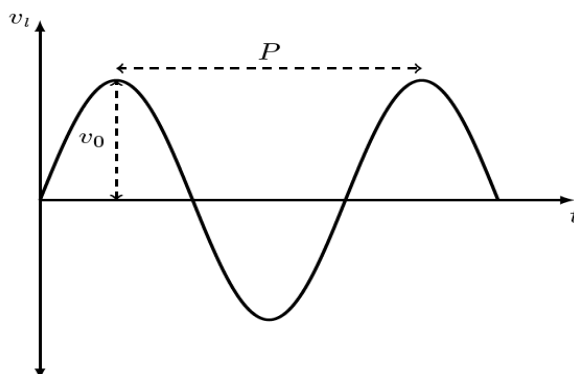
Os dois principais métodos de detecção de exoplanetas (ou seja, planetas em torno de outras estrelas) são o método da velocidade radial (ou “bamboleio”) e o método do trânsito. Neste problema nós iremos descobrir como a combinação dos resultados destes dois métodos pode trazer muita informação sobre a órbitas destes exoplanetas e de suas estrelas hospedeiras.

Ao longo deste problema, vamos considerar o caso de um planeta de massa M_p e raio R_p , movendo-se numa órbita circular de raio a em torno de uma estrela de massa M_s ($M_s \gg M_p$) e raio R_s . A normal ao plano da órbita do planeta é o ângulo de inclinação i em relação à nossa linha de visada ($i = 90^\circ$ significa órbita vista de perfil). Vamos assumir que não há outros planetas orbitando a estrela, e $R_s \ll a$.

Método da velocidade radial:

Quando um planeta e uma estrela orbitam em torno do baricentro do sistema, a estrela parece mover-se ligeiramente, ou oscilar, uma vez que o centro de massa da estrela não coincide com o baricentro do sistema estrela-planeta. Como consequência, a luz recebida da estrela sofre um pequeno deslocamento Doppler relacionado à velocidade desta oscilação.

A velocidade radial da estrela v_r pode ser determinada pelo deslocamento Doppler de uma linha espectral conhecida, e o período de sua variação é mostrado no diagrama esquemático abaixo. Neste diagrama, podemos identificar as duas quantidades mensuráveis neste método, que são o período orbital P e a velocidade radial máxima v_0 .



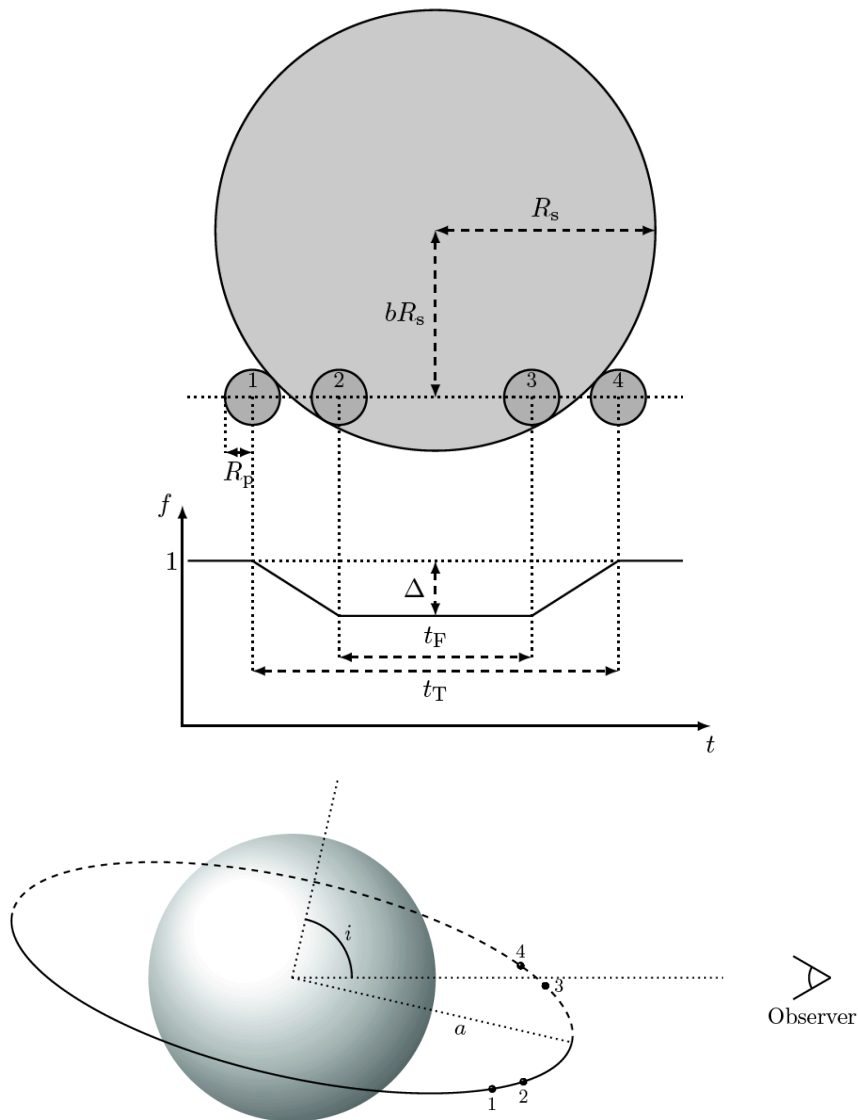
(T12.1) Encontre as expressões para o raio orbital (a) e a velocidade orbital (v_p) do planeta em função de M_s e P . 3

(T12.2) Obtenha o limite inferior para a massa do planeta $M_{p,\min}$ em função de M_s , v_θ e v_p . 4

Método do trânsito:

À medida que um planeta orbita sua estrela, para orientações do plano orbital vistas de perfil ($i \approx 90^\circ$), ele passa periodicamente, ou transita, na frente do disco estelar. Isto irá causar uma pequena diminuição do fluxo estelar observado, que pode ser medida. O esquema abaixo (fora de escala) mostra esta situação da perspectiva do observador, e a curva de luz do trânsito resultante (fluxo normalizado f vs tempo t) para um disco estelar de brilho uniforme.

Se o ângulo de inclinação i for exatamente de 90° , o planeta cruzará o disco estelar ao longo de seu diâmetro. Para outros valores de i , o trânsito ocorrerá ao longo de uma corda, cujo centro se encontra a uma distância bR_s do centro da estrela, como mostrado na figura a seguir. O fluxo fora do trânsito está normalizado para 1 e sua diminuição máxima é dada por Δ .



Os quatro pontos relevantes do trânsito são o primeiro, segundo, terceiro e quarto contatos, marcados na figura acima pelas posições de 1 a 4, respectivamente. O intervalo de tempo entre o segundo e terceiro contato é indicado como t_F , quando o disco do planeta passa inteiro em frente ao disco estelar. O intervalo

de tempo entre o primeiro e quarto contato é indicado por t_T . Esses pontos também estão indicados no diagrama esquemático acima, que mostra uma visão lateral da órbita (fora de escala).

As quantidades que podem ser medidas no método do trânsito são P , t_T , t_F e Δ .

(T12.3) Encontre o vínculo em i em função de R_s e a para que o trânsito seja visível para um observador distante. 2

(T12.4) Expresse Δ em função de R_s e R_p . 1

(T12.5) Expresse t_T e t_F em termos de R_s , R_p , a , P e b . 8

(T12.6) Na aproximação de um órbita muito maior que o raio estelar, mostre que o parâmetro b é dado por 5

$$b = \left[1 + \Delta - 2\sqrt{\Delta} \frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2} \right]^{1/2}$$

(T12.7) Use o resultado do item (T13.6) para obter uma expressão para a razão a/R_s em termos de parâmetros mensuráveis do trânsito, usando uma aproximação adequada. 3

(T12.8) Combine os resultados dos métodos da velocidade radial e do trânsito para determinar a densidade estelar média $\rho_s \equiv \frac{M_s}{4\pi R_s^3/3}$ em termos de t_T , t_F , Δ e P . 6

Rochoso ou gasoso:

Vamos considerar um sistema estrela-planeta visto de perfil ($i = 90^\circ$), com a órbita do planeta circular. A massa da estrela é $1.00M_\odot$. Os trânsitos são observados com período (P) de 50.0 dias e duração total t_T de 1.00 hora. A profundidade do trânsito (Δ) é 0.0064. O mesmo sistema também é observado pelo método de velocidade radial, onde exibiu uma velocidade máxima na linha de visada de 0.400 ms^{-1} .

(T12.9) Encontre o raio orbital a do planeta em UA e em metros. 2

(T12.10) Encontre a razão t_F/t_T do sistema. 2

(T12.11) Obtenha a massa M_p e o raio R_p do planeta, expressos em termos da massa ($M_\oplus$) e raio ($R_\oplus$) da Terra. Qual a composição mais provável do planeta, rochosa ou gasosa? Assinale a resposta ROCKY ou GASEOUS na Folha de respostas. 8

Curvas de luz de trânsitos realísticos:

(T12.12) Considere um trânsito com $i = 90^\circ$ de um planeta ao redor de uma estrela que tem uma mancha em seu equador, de tamanho comparável a tamanho do planeta de raio R_p . O período de rotação da estrela é $2P$. Esboce diagramas esquemáticos da curva de luz do trânsito para cinco trânsitos sucessivos do planeta, nos espaços apropriados da folha de respostas. O fluxo da estrela fora do trânsito pode ser normalizado para 1 independentemente em cada gráfico. Considere que o planeta não se sobrepõe à mancha no primeiro trânsito, mas o faz no segundo. 4

(T12.13) Nesta questão consideramos uma estrela com disco uniformemente brilhante. Entretanto, estrelas reais possuem discos com escurecimento de borda. Esboce uma curva de luz de trânsito onde o escurecimento de borda está presente na estrela. 2

QUESTÕES CURTAS

1. O sistema Gliese 876 ($M_G = 0.33 \pm 0.03 M_\odot$) possui diversos exoplanetas, listados abaixo:

Nome	Massa	Semi-eixo maior (UA)
Gliese 876 b	$2,276 M_J$	0,2083
Gliese 876 c	$0,714 M_J$	0,1296
Gliese 876 d	$6,8 M_\oplus$	0,0208
Gliese 876 e	$15 M_\oplus$	0,334

onde $M_\odot$ é a massa do Sol, M_J a massa de Júpiter ($M_J = 1,89813 \times 10^{27} \text{ kg}$), e $M_\oplus$ a massa da Terra. Considere que todos os exoplanetas orbitam Gliese 876 na mesma direção. Dizemos que dois planetas possuem órbitas ressonantes se o período sinódico de um planeta com relação ao outro planeta é um múltiplo inteiro do período orbital do segundo planeta. Descubra quais exoplanetas do sistema Gliese 876 possuem órbitas ressonantes.

2. Um satélite de um planeta tem período orbital de 7 dias, 3 horas e 43 minutos, e um semi-eixo maior da órbita que é 15,3 vezes maior que o raio médio do planeta. Nossa Lua possui período orbital de 27 dias, 7 horas e 43 minutos, e um semi-eixo maior da órbita que é 60,3 vezes maior que o raio médio da Terra. Calcule a razão entre as densidades médias do planeta e da Terra. Considere que a massa da Lua e do satélite são desprezíveis com relação às massas dos planetas.
3. Em 27 de Maio de 2015 às 02:18:49, a ocultação da estrela HIP 89931 (δ –Sgr) pelo asteroide 1285 Julietta foi observada do templo de Borobudur, localizado no centro da trajetória da sombra do asteroide. A duração foi de apenas 6,201 s. Suponha que a órbita da Terra é circular, que a órbita de Julietta encontra-se no mesmo plano da eclíptica, e que o sentido da sua órbita é o mesmo do da terrestre. No momento da ocultação, Julietta estava próxima do afélio. Sabe-se no instante da ocultação as distâncias de Julietta ao Sol e à Terra eram 3,076 UA e 2,156 UA, respectivamente. Encontre o diâmetro aproximado de Julietta, se o semi-eixo maior de sua órbita é $a = 2,9914$ UA.
4. Um observador tem à sua disposição um hipotético telescópio gigantesco, cujo espelho tem diâmetro igual ao da Terra, e que opera no infravermelho distante, na faixa de comprimentos de onda entre 20 a 640 μm . Com este telescópio, ele descobre um buraco negro supermassivo, estático e neutro, com massa $2,1 \times 10^{10} M_\odot$. Determine a distância máxima para qual este buraco negro pode ser resolvido com este telescópio.



5. Um observador tenta determinar o valor aproximado da excentricidade orbital de um satélite artificial. No momento do apogeu, verificou-se que o satélite move-se de um ângulo $\Delta\theta_1 = 2'44''$ em um curto intervalo de tempo. Foi verificado também que, quando o raio vetor que liga a Terra ao satélite é perpendicular ao eixo maior da órbita (isto é, a anomalia verdadeira é 90°), o satélite move-se de um ângulo $\Delta\theta_2 = 21'17''$ no mesmo intervalo de tempo. Encontre o valor aproximado da excentricidade da órbita do satélite (considere que o observador está localizado no centro da Terra).
6. Antes de cada sessão de observação, um rádio telescópio é apontado para uma fonte pontual de calibração, cuja densidade de fluxo medida fora da atmosfera terrestre é 21,86 Jy. Num certo dia, a densidade de fluxo da fonte de calibração medida pelo rádio telescópio foi 14,27 Jy. Se neste momento a altura da fonte era de 35 graus acima do horizonte, estime a profundidade ótica atmosférica no zênite, τ_z .
7. Uma galáxia, localizada na borda de um aglomerado de galáxias de raio 10 Mpc, conseguirá escapar do aglomerado se tiver velocidade inicial de pelo menos 700 km/s relativa ao centro do aglomerado. Calcule a densidade do aglomerado.
8. Um forte sinal contínuo de rádio vindo de um corpo celeste foi observado como uma explosão de curtíssima duração de 700 μ s. A densidade de fluxo observada na frequência de 1660 MHz foi de 0,35 kJy. Se a fonte estiver a uma distância conhecida de 2,3 kpc, estime a temperatura de brilho desta fonte.
9. Considere que o Sol e Vênus se comportam como corpos negros perfeitos. Vênus, que está a uma distância orbital de 0,72 UA, possui temperatura T_v , e está em equilíbrio térmico (isto é, irradia a mesma quantidade de energia que recebe do Sol). Suponha que, na sua maior aproximação da Terra, o diâmetro angular de Vênus é de 66 segundos de arco. Qual a densidade de fluxo de Vênus, no momento de sua maior aproximação da Terra, quando observado com um rádio telescópio na frequência de 5 GHz?
10. Uma nuvem molecular de hidrogênio possui temperatura $T = 115$ K. Os átomos desta nuvem (considerados esféricos) possuem raio (covalente) de $r_H = 0,37 \times 10^{-10}m$, e a separação entre o centro de dois átomos é $d_{H_2} = 0,74 \times 10^{-10}m$. Considere que as moléculas estão em equilíbrio térmico. Estime a frequência com que as moléculas irradiam devido à excitação molecular rotacional.

11. A densidade de massa de um objeto é inversamente proporcional à distância radial do seu centro, com um fator de proporcionalidade $\alpha = 5,0 \times 10^{13} \text{ kg/m}^2$. Calcule a massa total deste objeto, sabendo que sua velocidade de escape na superfície é $v_0 = 1,5 \times 10^4 \text{ m/s}$.
12. Um próton com energia cinética de 1 GeV está se propagando da superfície do Sol em direção à Terra. Desprezando o campo magnético do Sol, calcule o tempo de viagem deste próton como visto da Terra.
13. A atividade vulcânica de Io, cujo período de rotação é sincronizado com seu período orbital, foi proposta como o resultado do aquecimento devido, principalmente, à maré provocada por Júpiter. A força de maré resultante sobre um corpo é a diferença das forças gravitacionais experimentadas pelos lados próximos e distantes deste corpo, provocadas por outro corpo. Medidas das distorções da superfície de Io feitas por mapeamento de satélite com altímetro de radar indicam que a superfície sobe e desce em até 100 m durante meia órbita. Apenas as camadas superficiais se movem desta quantidade. As camadas internas se movem menos, de modo que assumiremos que, na média, toda a massa de Io se move por 50 m . Considere Io como composto por dois hemisférios, cada um deles tratado como uma partícula pontual. Calcule a *potência média* do aquecimento de maré devido à força gravitacional de Júpiter sobre Io.

Dica : use a aproximação $(1 + x)^n \approx 1 + nx$ para pequenos valores de x . São dados:

Massa de Io: $m_{Io} = 8,931938 \times 10^{22} \text{ kg}$

Perijove (distância mínima até Júpiter): $r_{peri} = 420000 \text{ km}$

Apojove (distância mínima até Júpiter): $r_{apo} = 423400 \text{ km}$

Período orbital de Io: 152853 s

Raio de Io: $R_{Io} = 1821,6 \text{ km}$,

Massa de Júpiter: $M_J = 1,89813 \times 10^{27} \text{ kg}$

14. Suponha que nós vivamos num universo estático e infinitamente grande, onde a densidade média de estrelas é $n = 10^9 \text{ Mpc}^{-3}$ e o raio médio das estrelas é igual ao raio solar. Suponha que a geometria euclidiana padrão é válida para esse universo. Quão longe, em média, poderíamos ver, em qualquer direção, antes de ter uma estrela na linha de visada? Dê sua resposta em Mpc.
15. Um avião voa de Lima, capital do Peru, ($12^\circ 2' S$ e $77^\circ 1' W$) para Yogyakarta ($7^\circ 47' S$ e $110^\circ 26' E$), local próximo à sede da 9ª IOAA. O avião escolhe a rota mais curta de Lima a Yogyakarta. Encontre a latitude do ponto mais ao sul desta rota.



QUESTÕES LONGAS

1. Uma lua orbita um planeta de maneira que o plano de sua órbita é perpendicular à superfície do planeta no local onde o observador está situado. Após aplicados os fatores de escala necessários, a órbita satisfaz a equação:

$$9\left(\frac{x}{2} + \frac{\sqrt{3}y}{2} - 4\right)^2 + 25\left(-\frac{\sqrt{3}x}{2} + \frac{y}{2}\right)^2 = 225$$

Considere coordenadas cartesianas, onde x está no plano horizontal e y no zênite do observador. Seja r o raio da lua. Assuma que o período de rotação do planeta é muito maior que o período orbital da lua. Ignore a refração atmosférica.

- a. Calcule os semi-eixos maior e menor da elipse.
- b. Calcule o ângulo zenital do perigeu.
- c. Determine $\tan \frac{\theta}{2}$, onde θ é o ângulo de elevação (altura da tangente superior da lua), quando a lua parece maior para o observador.

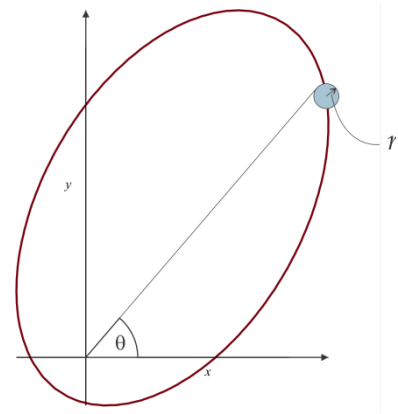


Figura 1

2. Duas estrelas massivas A e B, com massas m_A e m_B , estão separadas por uma distância d . Ambas estrelas orbitam o centro de massa do sistema. Suponha que as órbitas são circulares e que estão contidas no plano X-Y, cuja origem fica no centro de massa do sistema (Figura 2)

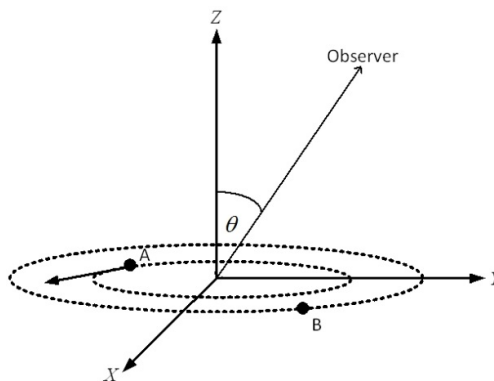


Figura 2

- a. Encontre as expressões para as velocidades tangencial e angular da estrela A.

Um observador localizado no plano Y-Z (Figura 2) observa as estrelas distantes com um ângulo θ relativo ao eixo Z. Ele mede a componente da velocidade da estrela A ao longo da linha de visada e conclui que ela possui a forma $K \cos(\omega t + \varepsilon)$, onde K e ε são constantes positivas.

- b. Expresse o valor de $K^3/\omega G$ em termos de m_A , m_B , e θ , onde G é a constante da gravitação universal.

Suponha agora que o observador consegue medir a massa da estrela A, e que essa vale $30M_S$, onde M_S é a massa do Sol. Além disso, ele observa que a estrela B emite raios-X, o que o leva a suspeitar que ela possa ser uma estrela de nêutrons ou um buraco negro. Esta conclusão depende do valor de m_B , da seguinte maneira: i) se $m_B < 2M_S$, então B é uma estrela de nêutrons; ii) se $m_B > 2M_S$, então B é um buraco negro.

- c. O observador realizou uma medição que forneceu $\frac{K^3}{\omega G} = \frac{1}{250}M_S$. Na prática, o valor de θ geralmente é desconhecido. Para quais valores de θ a estrela B pode ser um buraco negro?

3. Suponha que uma estrela esférica e estática é composta de N partículas neutras, e que seu raio é R (Figura 3).

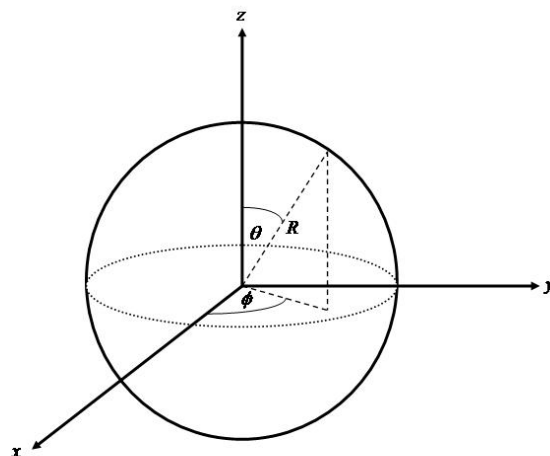


Figura 3



Sejam $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$, e a seguinte equação de estado é satisfeita

$$P V = N k \frac{T_R - T_0}{\ln(T_R/T_0)} \quad (1)$$

onde P e V são a pressão no interior da estrela e o volume da estrela, respectivamente, e k é a constante de Boltzmann. T_R e T_0 são as temperaturas na superfície ($r = R$) e no centro ($r = 0$), respectivamente. Suponha $T_R \leq T_0$.

- a. Simplifique a equação de estado (1) usando $\Delta T = T_R - T_0 \approx 0$. Este caso é chamado de "estrela ideal". (Dica: use a aproximação $\ln(1+x) \approx x$ para pequenos valores de x)

Suponha que a estrela passe por um processo quase-estático, no qual ela pode passar por pequenas contrações e expansões, mas sem invalidar a equação de estado (1).

A estrela satisfaz a Primeira Lei da Termodinâmica

$$Q = \Delta M c^2 + W \quad (2)$$

Onde Q , M , e W são o calor, massa da estrela, e trabalho, respectivamente, c é a velocidade da luz no vácuo, e $\Delta M \equiv M_{\text{final}} - M_{\text{inicial}}$.

No próximo item, iremos assumir T_0 constante, enquanto $T_R \equiv T$ é variável.

- b. Encontre a capacidade térmica da estrela a volume constante C_v em termos de M , e a capacidade térmica a pressão constante C_p em termos de C_v e T . (Dica: use a aproximação $(1+x)^n \approx 1 + nx$ para pequenos valores de x)

Suponha agora que C_v é constante e que o gás passa por um processo isobárico, de modo que a estrela produz calor e o irradia para o espaço.

- c. Encontre o calor produzido pelo processo isobárico, se as temperaturas inicial e final forem T_i e T_f , respectivamente.
- d. Suponha que um observador está muito distante da estrela. Usando as informações do item c, estime a distância do observador até a estrela.

Para os próximos dois itens, assuma que a estrela é o nosso Sol.

- e. Se a luz solar for monocromática, com frequência 5×10^{14} Hz, estime o número de fótons irradiados pelo Sol a cada segundo.
- f. Calcule a capacidade térmica C_v do Sol assumindo que sua temperatura superficial varia de 5500 K até 6000 K em um segundo.

Prova Teórica - Questões Curtas

- Qual seria a temperatura média da superfície da Terra, ignorando o efeito estufa, e considerando que seu albedo é constante? Suponha que a Terra é um corpo negro perfeito e que sua trajetória em torno do Sol é circular.
- Vamos supor que nós observamos um planeta tipo “Júpiter quente” orbitando uma estrela a uma distância média $d = 5$ UA. Este sistema está a $r = 250$ pc de nós. Qual o menor diâmetro D que um telescópio deve ter para ser capaz de observar estes dois objetos (planeta e estrela)? Assuma que a observação é feita na região visível do espectro eletromagnético ($\lambda \sim 500$ nm), fora da atmosfera da Terra, e que a óptica do telescópio é perfeita.
- Estima-se que o Sol levará cerca de $t_1 = 10$ bilhões de anos para evoluir e sair da sequência principal. Estime o tempo t_2 correspondente ao mesmo fenômeno, caso o Sol fosse 5 vezes mais massivo. Se necessário, utilize o gráfico da Figura 1.

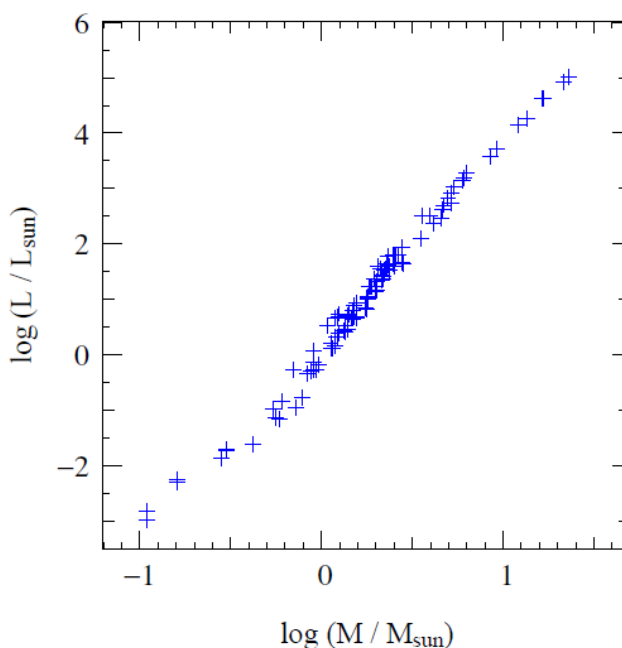


Figura 1: Relação Massa x Luminosidade para estrelas de sequência principal

- A Figura 2 mostra a relação entre a magnitude absoluta e o período para as Cefeidas clássicas. A Figura 3 mostra a curva de luz (magnitude aparente em função do tempo, em dias) de uma Cefeida clássica em uma galáxia do Grupo Local. (a) Utilizando estas duas figuras, estime a distância desta Cefeida até nós. (b) Revise sua estimativa supondo que a extinção interestelar na direção da Cefeida é $A = 0,25$ mag.

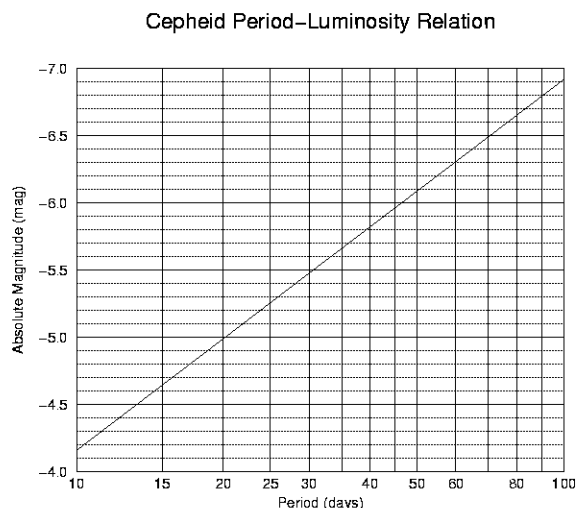


Figura 2

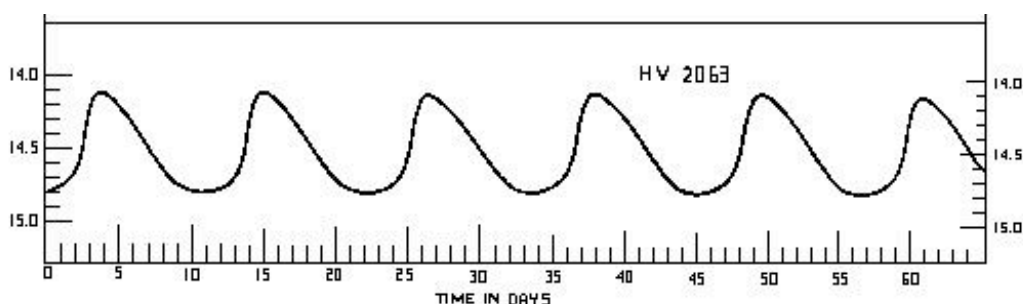
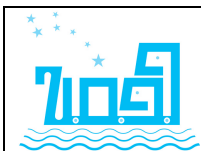


Figura 3

5. O espectro óptico de uma galáxia, cuja distância medida é de $41,67 \text{ Mpc}$, mostrou a linha $H\alpha$ de Balmer ($\lambda_0 = 656,3 \text{ nm}$) deslocada para $\lambda = 662,9 \text{ nm}$. (a) Utilizando estes dados, determine o valor da constante de Hubble H_0 . (b) A partir deste resultado, estime o tempo de Hubble para o Universo.
6. Uma estrela possui temperatura efetiva $T_{\text{eff}} = 8700 \text{ K}$, magnitude absoluta $M = 1,6$ e magnitude aparente $m = 7,2$. Encontre (a) a distância r até esta estrela; (b) sua luminosidade L ; e (c) seu raio R . Ignore os efeitos de extinção.
7. Uma estrela possui magnitude visual aparente $m_v = 12,2 \text{ mag}$, paralaxe $\pi = 0,001''$ e temperatura efetiva $T_{\text{eff}} = 4000 \text{ K}$. Sua correção bolométrica é $B.C. = -0,6 \text{ mag}$. (a) Encontre sua luminosidade em função da luminosidade solar. (b) Que tipo de estrela é esta: (i) uma gigante vermelha (ii); uma gigante azul; ou (iii) uma anã vermelha? Escreva (i), (ii) ou (iii) na sua folha de respostas.
8. Um sistema binário é composto pelas estrelas **a** e **b**. A razão de brilho entre elas é igual a 2. Este sistema é difícil de ser resolvido, e é observado da Terra com uma só estrela de 5^a magnitude. Encontre a magnitude aparente de cada uma das estrelas (m_a e m_b).
9. Encontre as coordenadas equatoriais (*ângulo horário* e *declinação*) de uma estrela, observada de Madri (latitude geográfica $\varphi = 40^\circ$) quando a estrela tem seu ângulo zenital $z = 30^\circ$ e azimute $A = 50^\circ$. O azimute é medido a partir do Sul.



10. No centro de nossa Galáxia, na intensa fonte de rádio Sgr A*, existe um buraco negro de grande massa. Uma equipe de astrônomos mediu a distância angular de uma estrela a Sgr A* e seu período orbital em torno dela. A distância angular foi de $0,12''$ (segundos de arco) e o período de 15 anos. Calcule a massa deste buraco negro em *massas solares*, considerando que a órbita da estrela é circular.
11. Qual é a altitude máxima, a_M (max), em que a Lua Cheia pode ser observada de Thessaloniki? A latitude geográfica de Thessaloniki é $\varphi_\Theta = +40^\circ 37'$. Leve em consideração todos os fatores possíveis.
12. *Sirius A*, com magnitude visual $m_V = -1,47$ (a estrela mais brilhante do céu) e raio estelar $R_A = 1,7R_\odot$, é a estrela primária de um sistema binário. A existência de sua companheira, *Sirius B*, foi deduzida através da astrometria em 1844, pelo famoso matemático e astrônomo Friedrich Bessel, antes dela ser observada diretamente. Assumindo que ambas as estrelas são do mesmo tipo espectral e que *Sirius B* é 10 magnitudes menos brilhante que *Sirius A* ($\Delta m = 10$), calcule o raio estelar de *Sirius B*.
13. Graças a uma espessa camada de neblina, o Sol foi recentemente observado em Londres com uma magnitude visual igual à da Lua Cheia quando observada em uma noite de céu limpo. Considerando que a redução na intensidade da luz devido à neblina é dada por uma relação exponencial, calcule o expoente τ , mais conhecido como profundidade óptica.
14. Qual o ângulo horário H e o ângulo zenital z de Vega ($\delta = +38^\circ 47'$) quando observada desde Thessaloniki ($\lambda_1 = +1^h 32^m$, $\varphi_1 = +40^\circ 37'$), no mesmo momento em que ela faz a passagem meridiana em Lisboa ($\lambda_2 = -00^h 36^m$, $\varphi_2 = +39^\circ 43'$)?
15. O desvio Doppler de três galáxias distantes foi medido com o auxílio de observações espectroscópicas. Os valores estão na Tabela abaixo.

Galaxy	Redshift (z)
3C 279	0,536
3C 245	1,029
4C41.17	3,8

- (a) Calcule a velocidade aparente de recessão de cada galáxia (1) usando física clássica; (2) usando a aproximação $v = c \ln(1+z)$, frequentemente utilizada em cosmologia; e (3) usando a relatividade restrita.
- (b) Para os três casos, a que porcentagem da velocidade da luz as galáxias aparentam se afastar?

(T1) Super Luminal Galaxies

(10 points)

Read the statements given below and state if they are true or false:

- (a) For some galaxies the apparent recession speed exceeds the speed of light.
- (b) The velocity – Distance relation as given by Hubble cannot allow recession velocities to exceed the speed of light.
- (c) Hubble-Lemaitre's law (formerly known as Hubble's Law) does not violate special relativity.
- (d) If some galaxies would have an apparent recession speed exceeding the speed of light, then the photons from those galaxies can never reach us.
- (e) As the expansion of Universe is accelerating, photons emitted right now from galaxies which have apparent recession speed equal to the speed of light will never reach us.

Solution:

T F T F F (2 points each)

(T2) Distance points)

(10

An observer measured trigonometric parallaxes of stars in a star cluster. Due to random errors, the measured parallax values are distributed symmetrically around the expected value with standard deviation equal to 0.05 mas (milliarcsec). Assume there are no systematic errors and assume all stars in the said cluster have the same luminosity. It is known that the distance of this cluster from us is $R = 5$ kpc.

He gave the data table to 4 of his students (A, B, C and D) and they estimated the distance to the cluster in the following ways:

- A. Convert each parallax measurement into distance and then find the average distance (R_A)
- B. Take the average of all parallaxes first and then convert the average parallax into distance. (R_B)
- C. Convert each parallax measurement into distance and then take the median distance value. (R_C)
- D. Find the median value of the parallaxes and then convert the median value into distance. (R_D)

State if the following statements are true or false. **In case a given mathematical relation is false, give the correct relation.**

(l) If the i^{th} star gave the smallest value of parallax and the j^{th} star gave the highest value of parallax, in all likelihood $R_i - R > R - R_j$

- (m) $R_A = R$ (i.e. there is a high chance that the distance estimated by A fairly matches the true distance)
- (n) $R_B = R$ (i.e. there is a high chance that the distance estimated by B fairly matches the true distance)
- (o) $R_C < R$ (i.e. there is a high chance that the distance estimated by C will be systematically lower than the true distance)
- (p) $R_D = R$ (i.e. there is a high chance that the distance estimated by D fairly matches the true distance)

Solution:

T F T F T (2 points each)

For parallaxes, we have a Gaussian with $\mu = 0.2$ mas and $\sigma = 0.05$ mas i.e. 25% of the expectation value.

If we assume that the lowest and highest values of parallax are symmetric w.r.t. the expected value (reasonable assumption if cluster has large number of stars), then their distances would not be symmetric w.r.t. the true distance.

$R_A > R$, $R_C = R$ (for statement m and o, marks will be given only if this relation is stated)

(T3) Atmospheric Refraction

(10 points)

Consider sunrise at Beijing ($\phi=40^\circ$) on the vernal equinox day.

(a) Let us say r_l , r_d , r_r and r_u are distances from the centre of the undistorted disk of the Sun to the edge of the disk towards the directions left, down, right and up respectively. What will be the hierarchical relation ($<$, $=$, $>$) between the four radii just after the sunrise?

(b) What is the correction in the time of rise of the top edge of the disk as compared to the case without atmosphere? You may assume that typically atmospheric refraction near the horizon is $35'$. Please only consider the apparent diurnal motion.

Solution:

(a) $r_d < r_u < r_l = r_r$

(4 points)

(b) When the upper edge of the sun is under the horizon with $35'$, it will appear on the horizon earlier due to the influence of atmospheric refraction. With the apparent diurnal motion, i.e.

$$\omega = \frac{360^\circ}{23^h 56^m 4^s} = \frac{21600'}{1436.0682^m} = 15.0411' / m, \quad (2 \text{ points})$$

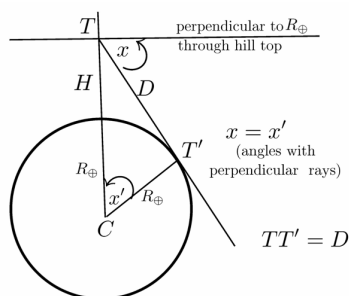
Ideally one should find time taken by mean sun to travel from $50'$ below horizon to $15'$ below true horizon.

$$\Delta t = \frac{(50' - 15')}{\omega \cos \phi} = \frac{35'}{15.0411' \times \cos 40^\circ} = 3.04 \text{ mins} \quad (4 \text{ points})$$

(T4) Height of a Hill

(10 points)

Two friends wanted to measure the height of the hill next to their village (latitude $\phi=40^\circ$). One of the friends climbed to the top of the hill and she agreed to send a light signal to her friend in the village as soon as she sees the sunset. On March 21, when they did this experiment, the friend in village received the light signal 4.1 minutes after the sunset from the village. Estimate the height of the hill and horizon distance for the person at the hill top. Ignore atmospheric refraction.



Solution:

First we should realise that the Sun doesn't set vertically. It sets at an angle of $\theta = 90 - \varphi = 50^\circ$. Thus, the horizon depression x will be

$$x = 15.0411' \times 4.1 \times \sin\theta = 1.03^\circ \sin 50^\circ = 0.787^\circ. \quad (4 \text{ points})$$

Let the height of the mountain be H . Let C be the centre of the Earth, point T - hill top and point T' is the point on the horizon as seen from T . Thus,

$$\cos x = \left[\frac{R_\oplus}{(R_\oplus + H)} \right]$$

$$H = \frac{R_\oplus(1 - \cos x)}{\cos x} = 602 \text{ metres} \quad (3 \text{ points})$$

The distance of horizon (from base of the mountain, along the surface of the earth) is found from the rectangular triangle TCT' as

$$D = (R_\oplus + H)\sin x \approx R_\oplus \sin x = 6.38 \times 10^3 \sin 0.787^\circ = 87.6 \text{ km}$$

$$\text{Alternatively, } D = R_\oplus x = 6.38 \times 10^3 \times 0.787^\circ \times \pi \div 180^\circ = 87.6 \text{ km} \quad (3 \text{ points})$$

(T5) Sidereal Time

(10 points)

It is very interesting to observe that on one particular calendar day each year, the mean sidereal time will twice be 00:00:00.

- (a) What will be the approximate R.A. of the Sun when this event happens?
- (b) Estimate the exact date in 2018 for this event.

You may assume that at the Royal Greenwich Observatory, the mean sidereal time (GMST_0) was 6.706h at 0h, 1st January, 2018 (JD2458119.5).

Solution:

(1) This can happen if the sidereal time is 00:00:00 just after the midnight and just before the midnight on the same day. It means that the sun should be near the Autumn (September) Equinox. Approximate R.A. of the Sun is 18^{h} . (3 points)

(2) At 0^{h} , 1st January, 2018 (JD2458119.5), the mean sidereal time GMST_0 was 6.706^{h} , i.e. $\text{GMST}_0 = 100.59^\circ$. Since one mean solar day is longer than one mean sidereal day with 0.9856° (about 4 minutes), the GMST will increase $\Delta = 0.9856^\circ$ at next midnight at 2nd January, 2018. Therefore, we can first calculate how many days it will take for GMST to reach 360° , i.e.

$$T = \frac{360^\circ - \text{GMST}_0}{\Delta} = \frac{360^\circ - 100.59^\circ}{0.9856^\circ} = 263.20 \quad (3 \text{ points})$$

i.e. after completion of 263 days, the GMST at midnight will be 359.80° .

Thus, GMST will be 00:00:00 on 264th day at 00:00:48 and then again at 23:56:52 on the same day.

(2 points)

The 264th day in the calendar of 2018 is 21st September 2018.

(2 points)

(T6) Observe the Sun with FAST

(25 points)

The Five-hundred-meter Aperture Spherical radio Telescope (FAST) is a single-dish radio telescope located in Guizhou Province, China. The physical diameter of the dish is 500 m, but during observations, the effective diameter of the collecting area is 300 m.

Consider observations of the thermal radio emission from the photosphere of the Sun at 3.0 GHz with this telescope and a receiver with bandwidth 0.3 GHz.

- Calculate the total energy ($E_{\odot}$) that the receiver will collect during 1 hour of observation.
- Estimate the energy needed to turn over one page of your answer sheet (E'). Hint: the typical surface density of paper is 80 gm^{-2} .
- Which one is larger?

Solution:

Rayleigh-Jeans law to calculate the thermal emission from the sun at 3 GHz states,

$$B_{\nu} = \frac{2k_B T}{c^2} \nu^2$$

which is the power emitted per unit emitting area, per steradian, per unit frequency. Therefore, the solar luminosity at 3 GHz should be:

$$L_{\nu} = B_{\nu} \cdot 4\pi R_{\odot}^2 \quad (3 \text{ points})$$

At the distance of the earth (1 AU), the monochromatic flux from the sun at 3 GHz should be:

$$f_{\nu} = \frac{L_{\nu}}{4\pi D^2}$$

Hence the energy flux that FAST will receive is:

$$F_{\nu} = f_{\nu} \cdot \Delta\nu \cdot \pi^2 \frac{d_{\text{eff}}^2}{4} \quad (8 \text{ points})$$

And the total energy that the receiver will collect during 1 hour of observation is:

$$E_{\odot} = F_{\nu} \Delta t = \frac{2k_B T}{c^2} \nu^2 \frac{R_{\odot}^2}{D^2} \cdot \pi^2 \frac{d_{\text{eff}}^2}{4} \Delta\nu \Delta t = 8.5 \times 10^{-5} \text{ J} \quad (8 \text{ points})$$

Then we can calculate the work we need to turn over one piece of the answer sheet (A4 paper). The mass of an A4 paper (297mm×210mm) is:

$$m = \rho \cdot L_1 L_2 \quad (2 \text{ point})$$

Therefore, the energy of turning it over should be around:

$$E' = mg \cdot \frac{L_2}{2} \approx 5 \times 10^{-3} \text{ J} \quad (3 \text{ points})$$

As a consequence, $E' > E_{\odot}$.

(1 point)

(T7) Sunspot

(25 points)

Magnetic fields are important in the physics of stars and sunspots. As an approximation, we can model the photosphere of the Sun consisting of a plasma, which can be simply treated as a single component ideal gas, and a magnetic field ($\mathbf{B}$), which has an associated magnetic pressure $p_B = \frac{B^2}{2\mu_0}$. It behaves like any other physical pressure except that it is carried by the magnetic field rather than by the kinetic energy of particles.

Assume that the number density of particles in the photosphere is constant everywhere, but the magnetic field inside the sunspot ($B_{\text{in}}=0.1\text{ T}$) is much stronger than outside ($B_{\text{out}}=5\times 10^{-3}\text{ T}$). From the blackbody spectrum, the temperature inside the sunspot is $T_{\text{in}}\sim 4000\text{ K}$, while the temperature outside is $T_{\text{out}}\sim 6000\text{ K}$ (which is why the sunspot looks darker). For the sunspot to be stable, the inside must be in equilibrium with the outside.

- (a) Estimate the number density of plasma particles in the solar photosphere.
- (b) Compare your answer with an estimate of the number density of particles in the atmosphere at the surface of the Earth.

Solution:

- (a) The pressure is from all directions. The plasma inside the sunspot must have the same total pressure as the plasma outside to maintain equilibrium on the border.

The kinetic pressure inside the sunspot is $p_i = n_i k_B T_i$, where n_i is the number density inside the sunspot.

And also, $p_e = n_e k_B T_e$. (3 points)

($p_e V = N_e k_B T_e$ is also correct. But it's better to write the pressure in terms of number density)

From the assumption, $n_i = n_e = n$. The equilibrium requires: (3 points)

$$n k_B T_i + \frac{B_i^2}{2\mu_0} = n k_B T_e + \frac{B_e^2}{2\mu_0} \quad (8 \text{ points})$$

Then

$$n = \frac{1}{2\mu_0 k_B} \frac{(B_i^2 - B_e^2)}{(T_e - T_i)} = 1.43 \times 10^{23} \text{ m}^{-3} \quad (6 \text{ points})$$

- (b) On the Earth, $p_E = n_E k_B T_E$ where $P_E \approx 10^5 \text{ Pa}$ and $T_E \approx 300 \text{ K}$.

$$\text{Thus, } n_E = \frac{p_E}{k_B T_E} = 2.4 \times 10^{25} \text{ m}^{-3} \quad (4 \text{ points})$$

This means that the number density of the atmospheric particles at the surface of the earth is at least 100 times larger than the number density of particles in the solar photosphere (of course the mass density is even higher). (1 point)

(T8) A Possible Dark Matter Deficient Galaxy

(25 points)

Earlier this year, a team of astronomers reported their discovery of a galaxy with much less dark matter than the galaxy evolution model predicted (van Dokkum et al. 2018, Nature). This galaxy, named NGC 1052-DF2, is located close to the elliptical galaxy NGC 1052 ($D=20\text{ Mpc}$ from the Sun) in the sky. The shape of NGC 1052-DF2 resembles an ellipse with semi major axis (a) of $22.6''$ and $\frac{b}{a} = 0.85$. Half of the total light from the

galaxy comes from within this ellipse and the mean surface brightness within the ellipse is about 24.7 mag arcsec⁻².

(a) Calculate the total apparent magnitude of this galaxy.

(b) The team suggested the galaxy is a companion of NGC 1052. Determine the total mass of stars in NGC 1052-DF2, assuming it has a mass to light ratio $\left(\frac{M/M_{\odot}}{L/L_{\odot}}\right)$ of 2.0.

(c) The team identified 10 globular clusters in NGC 1052-DF2 with a mean galactocentric distance of 78.4". They also measured the velocity dispersion of these clusters to be not more than 8.4 km/s. Estimate the dynamical mass of this galaxy. For simplicity, assume the mass distribution in the galaxies is uniform and is spherically symmetric.

(d) This discovery was challenged by other groups (Kroupa et al., Nature, 2018, Trujillo et al., MNRAS, 2018), who claimed that NGC 1052-DF2 is not a satellite of NGC 1052, and it is located at a much smaller distance to us. Show why a smaller distance would weaken the assertion of the dark matter deficiency in NGC 1052-DF2.

Solution:

(1) Total area within the ellipse (actually, half-light ellipse) of NGC 1052-DF2 is:

$$A_{DF2} = \pi ab = 0.85\pi \times 22.6''^2 = 1363.9 \text{ arcsec}^2 \quad (2 \text{ points})$$

Magnitude of the part of the galaxy within the ellipse:

$$m_{ell} = SB - 2.5 \times \log(A_{DF2}) = 24.7 - 2.5 \times \log(1363.9) = 16.9 \quad (2 \text{ points})$$

Total magnitude of the whole galaxy:

$$m_{gal} = m_{ell} - 2.5 \times \log(2) = 16.1 \quad (2 \text{ points})$$

(2) Absolute mag of the galaxy:

$$M_{0,gal} = m_{gal} + 5 - 5 \times \log(D) = -15.4 \quad (2 \text{ points})$$

Convert to solar luminosity:

$$\frac{L_{gal}}{L_{\odot}} = 10^{-0.4(M_{0,gal} - M_{0,\odot})} = 1.2 \times 10^8 \quad (2 \text{ points})$$

Thus, the total stellar mass should be:

$$\frac{M_{gal}}{M_{\odot}} = \frac{L_{gal}}{L_{\odot}} \times \frac{M}{L} = 2.4 \times 10^8 \quad (2 \text{ points})$$

(3) Mean galactocentric distance of globular clusters:

$$r_{gc} = \frac{\theta D}{206265} = 7.6 \text{ kpc} \quad (2 \text{ points})$$

Using virial theorem:

$$\langle 2K \rangle + \langle U \rangle = 0,$$

$$\langle K \rangle = \frac{1}{2} M \langle v^2 \rangle = \frac{1}{2} M \sigma^2,$$

$$\langle U \rangle = \frac{3}{5} \frac{GM^2}{r} \quad (4 \text{ points})$$

Thus the maximal dynamical mass of this system should be:

$$M_{dyn} = \frac{5r_{gc}\sigma^2}{3G} = 2.1 \times 10^8 M_{\odot}, \text{ even less than the total stellar mass within this radius.} \quad (3 \text{ points})$$

(4) From Question 2 and 3, we know:

$$M_{star} \propto L_{gal} \propto F_{gal} \times D^2$$

However, $M_{dym} \propto D$

(2 points)

Thus

$$M_{dym}/M_{star} \propto 1/D$$

(2 points)

If the distance measured is smaller by a certain factor, the stellar mass to dynamical mass ratio would increase by the same factor, thus the dark matter in NGC 1052-DF2 would not be as deficient as Dragonfly team claimed.

(T9) Radio Galaxy

(25 points)

An observer wants to use the Five-hundred-meter Aperture Spherical radio Telescope (FAST) in China to observe a radio galaxy at redshift of $z = 0.06$. We assume that the radio source is compact compared to the beam size of the telescope at the observing frequencies, i.e., the source is point-like as seen through the telescope. To detect a point source with FAST, it must be sufficiently strong (bright) relative to the noise level (for single polarization observations), σ , which depends on the bandwidth, $\Delta\nu$, and the integration time (the radio astronomy equivalent of exposure time), t_i , as follows:

$$\sigma = \frac{2k_B T_{sys}}{A_e \sqrt{t_i \Delta\nu}}$$

where T_{sys} is the system temperature (about 150 K in the frequency range of 0.28 GHz – 0.56 GHz and 25 K in the frequency range of 1.05 GHz – 1.45 GHz), and $A_e = 4.6 \times 10^4 \text{ m}^2$ is the effective area of the telescope taking into account the total efficiency of the instrument.

This radio galaxy has an observed continuum flux density of $f_\nu = 2.5 \times 10^{-3} \text{ Jy}$ at an observing frequency of 0.4 GHz. The bandwidth $\Delta\nu$ for the continuum observation centered at 0.4 GHz is $2.8 \times 10^8 \text{ Hz}$.

(a) In order to detect the continuum flux density at 0.4 GHz with a signal-to-noise ratio of 30 (a so-called 30σ detection), what is the required integration time, t_i ?

(b) We want to search for the neutral Hydrogen (HI) in the galaxy using 21cm absorption line. The HI 21cm line, with rest frame frequency of 1.4204 GHz. Calculate the observed frequency (ν_{obs}) of the HI line for this galaxy.

(c) The radio continuum emission from this galaxy can be described by a power law $f_\nu \sim \nu^\alpha$, with a spectral index of $\alpha = -0.2$. Calculate the continuum flux density at ν_{obs} for this galaxy.

(d) The line width of the HI 21cm absorption line is 90 km/s. Calculate the line width in Hz at the observing frequency of ν_{obs} . According to Figure 1, the HI 21cm line absorbs 4% of the continuum flux density (on average) over the line width of 90 kms^{-1} . In order to detect the absorption line at $\geq 3\sigma$ in three consecutive 30 kms^{-1} channels, what is the required integration time?

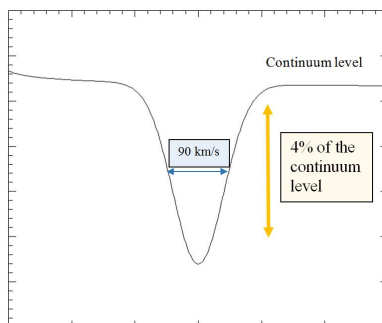


Figure 1: Spectrum of the HI 21cm absorption relative to the continuum emission in the radio galaxy

Solution:

(a) for signal to noise ratio of 30 of flux density of 2.5×10^{-3} Jy,

$$\sigma = 2.5 \times 10^{-3} / 30 \text{ Jy} \approx 8.33 \times 10^{-5} \text{ Jy} = 8.33 \times 10^{-31} \text{ Wm}^{-2}\text{Hz}^{-1} \quad (4 \text{ points})$$

$$T_{\text{sys}} = 150 \text{ K}, \Delta\nu = 280 \text{ MHz},$$

Using the equation above, $t_{\text{int}} \sim 42 \text{ s}$ (more exactly 41.8 s) (3 points)

(b) $\nu_{\text{obs}} = 1.4204 \text{ GHz} / (1+z) = 1.4204/1.06 = 1.34 \text{ GHz}$ (3 points)

(c) $f_{\nu_{\text{obs}}=1.34\text{GHz}} = f_{0.4\text{GHz}} \left(\frac{1.34}{0.4}\right)^{-0.2} = 2.5 \times 10^{-3} \text{ Jy} \times \left(\frac{1.34}{0.4}\right)^{-0.2} = 1.96 \times 10^{-3} \text{ Jy}$ (3 points)

(d) Line width : $90 \text{ kms}^{-1} / c \times \nu_{\text{obs}} = 90 / (2.9979 \times 10^5) \times 1.34 \text{ GHz} = 0.402 \text{ MHz}$ (2 points)

$$4\% \text{ of } 1.96 \times 10^{-3} \text{ Jy continuum} = 7.84 \times 10^{-5} \text{ Jy} \quad (1 \text{ point})$$

$$\geq 3\sigma \text{ means } \sigma = 7.84 \times 10^{-5} \text{ Jy} / 3 = 2.61 \times 10^{-5} \text{ Jy} = 2.61 \times 10^{-31} \text{ Wm}^{-2}\text{Hz}^{-1} \text{ (values around } 2.6 \times 10^{-5} \text{ Jy are ok)}$$

$$\text{in three consecutive } 30 \text{ kms}^{-1} \text{ channels: } \Delta\nu = 0.402 \text{ MHz}/3 = 0.134 \text{ MHz} \quad (6 \text{ points})$$

$$T_{\text{sys}} = 25 \text{ K}$$

Using equation (1), $t_{\text{int}} = 24700 \text{ s} \approx 6.9 \text{ hours}$ (3 points)

(T10) Vega and Altair

(75 points)

As per a very famous Chinese folklore about love, Vega and Altair are two lovers. It is said that they can meet each other once every year on a bridge made up of birds over the Milky Way. The parameters of two stars are given in the table below. For the purpose of this question, assume that the coordinate frame is fixed (i.e. not affected by precession or motion of the Sun).

Star	Right Ascension (J2000.0)	Declination (J2000.0)	Parallax (mas)	Proper Motion		Radial Velocity (km/s)
				$\mu_{\alpha} \cos \delta$ (mas/year)	μ_{δ} (mas/year)	
Vega	18 ^h 36 ^m 56.49 ^s	+38° 47' 07.7"	130.23	+200.94	+286.23	-13.9
Altair	19 ^h 50 ^m 47.70 ^s	+8° 52' 13.3"	194.95	+536.23	+385.29	-26.1

Based on this data, answer the following questions:

(a) (9 points) What is the angular separation of the two stars?

(b) (6 points) Calculate the distance (in parsecs) between Vega and Altair.

(c) (3 points) Calculate position angles of the proper motion vectors of each of these two stars.

For parts d-g, assume that the angular velocity of the stars on the celestial sphere remains constant. This is not a physical situation but this is an assumption to simplify the problem.

(d) (2 points) How many common points on the celestial sphere are there which can be reached by both these stars?

(e) (20 points) Find the coordinates of the closest such point.

(Note: Drawing the situation on a celestial sphere will help you in visualising the situation)

(f) (8 points) Find when (which year) each of these stars were / will be at that point.

(g) (5 points) When Altair was / will be at that point, what would be its angular separation from Vega?

(h) (22 points) Find coordinates of any point (if it exists) in 3-D space which was /will be visited by both these stars. Do not ignore radial velocities for this part of the question.

Solution:

(a) Assuming the positions of the two stars are: Vega: (α_1, δ_1) ; Altair : (α_2, δ_2)

$$(\alpha_1, \delta_1) = (279.23538^\circ, 38.78547^\circ) \text{ and } (\alpha_2, \delta_2) = (297.69875^\circ, 8.87036^\circ) \quad (2 \text{ points})$$

Cosine formula of the spherical geometry is given by:

$$\cos[a] = \cos[b]\cos[c] + \sin[b]\sin[c]\cos[A] \quad (0.5 \text{ point})$$

In our triangle, $b = (90 - \delta_1)$, $c = (90 - \delta_2)$ and $A = (\alpha_2 - \alpha_1)$. Then the angular distance β between Vega and Altair is:

$$\cos[\beta] = \cos[90^\circ - \delta_1]\cos[90^\circ - \delta_2] + \sin[90^\circ - \delta_1]\sin[90^\circ - \delta_2]\cos[\alpha_2 - \alpha_1]; \quad (3 \text{ points})$$

$$\cos[\beta] = \cos[51.21453^\circ]\cos[81.12964^\circ] + \sin[51.21453^\circ]\sin[81.12964^\circ]\cos[18.46337^\circ]; \quad (1.5 \text{ points})$$

$$\beta = 34.19582^\circ \quad (2 \text{ points})$$

(b) The distance of Vega is: $r_1 = 1/(130.23 \times 10^{-3}) = 7.6787pc$

The distance of Altair is: $r_2 = 1/(194.95 \times 10^{-3}) = 5.1295pc$ (2 points)

Using the Law of Cosines: $d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos[\beta]$ (2 point)

we can obtain the distance: $d = 4.4855pc$ (2 points)

(c) Given the proper motions and radial velocities of the two stars, the directions of their movements on the celestial sphere can be estimated.

For Vega:

$$\begin{aligned} \mu_{\alpha 1} \cos \delta_1 &= 200.94; \mu_{\delta 1} = 286.23 \\ \theta_1 &= \arctan \left[\frac{\mu_{\alpha 1} \cos \delta_1}{\mu_{\delta 1}} \right] \times \frac{180^\circ}{\pi} = 35.0697^\circ \end{aligned} \quad (2 \text{ points})$$

For Altair:

$$\begin{aligned} \mu_{\alpha 2} \cos \delta_2 &= 536.23; \mu_{\delta 2} = 385.29 \\ \theta_2 &= \arctan \left[\frac{\mu_{\alpha 2} \cos \delta_2}{\mu_{\delta 2}} \right] \times \frac{180^\circ}{\pi} = 54.302^\circ \end{aligned} \quad (1 \text{ points})$$

(d) As the position angles of the proper motion of two stars is different, the stars' paths will intersect. As the paths are circular on celestial sphere, they will intersect in exactly **two** points. (2 points)

(e) Let the closer point of intersection be I with the coordinates (α_3, δ_3) .

$$\text{In triangle PVA, } PAV = \arcsin \left[\frac{\sin VPA \sin PV}{\sin VA} \right] = \arcsin \left[\frac{\sin(\alpha_1 - \alpha_2) \sin(90^\circ - \delta_1)}{\sin \beta} \right] = 26.055^\circ$$

$$PVA = \arcsin \left[\frac{\sin VPA \sin PA}{\sin VA} \right] = \arcsin \left[\frac{\sin(\alpha_1 - \alpha_2) \sin(90^\circ - \delta_2)}{\sin \beta} \right] = 146.196^\circ \quad (5 \text{ points})$$

Note that angle PVA is more than 90° , which will be evident from the diagram. (1 point)

The motion of the stars on the celestial sphere will always be along some great circle. So VAI is a spherical triangle. Using four parts formula for VAI

$$\begin{aligned} \cot VI &= \frac{\cos VAcos IVA + \sin IVA \cot VAI}{\sin VA} \\ \cot VI &= \frac{\cos VAcos IVA + \sin IVA \cot VAI}{\sin VA} \end{aligned}$$

$$VI = 90^\circ - \arctan \left[\frac{\cos 34.19582^\circ \cos(68.900^\circ) + \sin(68.900^\circ) \cot(99.643^\circ)}{\sin 34.19582^\circ} \right] = 76.085^\circ$$

(5 points)

by sine rule, $AI = \arcsin \left[\frac{\sin VI \sin IVA}{\sin VAI} \right] = \arcsin \left[\frac{\sin 76.085^\circ \sin 68.900^\circ}{\sin 99.643^\circ} \right] = 66.715^\circ$

(2 points counted in next part)

Now we use triangle PVI

by cosine rule, $\cos[PI] = \cos[PV]\cos[VI] + \sin[PV]\sin[VI]\cos[PVI];$

$$\cos[90^\circ - \delta_3] = \cos[90^\circ - \delta_1]\cos[VI] + \sin[90^\circ - \delta_1]\sin[VI]\cos[180^\circ - \theta_1];$$

$$\sin[\delta_3] = \cos[51.215^\circ]\cos[76.085^\circ] + \sin[51.215^\circ]\sin[76.085^\circ]\cos[144.930^\circ];$$

Thus, $\delta_3 = -27.945^\circ$ (4 points)

Applying sine rule, $VPI = \arcsin \left[\frac{\sin PVI \sin VI}{\sin PI} \right] = \arcsin \left[\frac{\sin 144.930^\circ \sin 76.085^\circ}{\sin 117.945^\circ} \right] = 39.148^\circ$

Hence, $\alpha_3 = \alpha_1 - VPI = 279.235^\circ - 39.148^\circ = 240.087^\circ \approx 16^h 0^m 21^s$ (5 points)

Out of 5 points, 1 point is reserved for realising that the intersection point is in past path.

(f) The total velocity along the great circle for Vega is

$$\mu_1 = \sqrt{(\mu_{\alpha 1} \cos \delta_1)^2 + \mu_{\delta 1}^2} = \sqrt{200.94^2 + 286.23^2} = 349.72 \text{ mas/year} \quad (2 \text{ points})$$

$$\text{Number of years} = \frac{VI}{\mu_1} = \frac{76.085 \times 3600}{0.34972} \approx 783200 \text{ years}$$

It will happen in the year 781200 BCE (2 points)

Similarly, $\mu_2 = 660.30$ and number of years = 363700 years

It will happen in the year 3617 BCE (2 points)

(g) After 363700 years, Vega would have traversed 35.335° along its path. (3 points)

Thus, its separation from Altair will be $(76.085^\circ - 35.335^\circ) = 40.750^\circ$ (2 points)

(h) In earth centric Cartesian frame, coordinates of Vega (in units of parsec) will be:

$$x_1 = r_1 \cos \delta_1 \cos(360^\circ - \alpha_1) = 0.96062 \text{ pc}$$

$$y_1 = r_1 \cos \delta_1 \sin(360^\circ - \alpha_1) = 5.90793 \text{ pc}$$

$$z_1 = r_1 \sin \delta_1 = 4.80998 \text{ pc}$$

Similarly, coordinates of Altair will be, $x_2 = 2.35579$, $y_2 = 4.48736$ pc and $z_2 = 0.79097$ pc. (6 points)

Here, we are assuming the North as positive x-direction and pole as positive z-direction.

The direction vectors for velocities in spherical coordinates are

$$v_1 = (-13.9, 7.33, 10.45) \text{ and } v_2 = (-26.9, 19.57, 14.06) \quad (4 \text{ points})$$

The same in Cartesian coordinates will be

$$v_{x1} = v_{r1} \cos \delta_1 \cos \alpha_1 - v_{\alpha 1} \sin \alpha_1 - v_{\delta 1} \sin \delta_1 \cos \alpha_1 = 4.45$$

$$v_{y1} = -v_{r1} \cos \delta_1 \sin \alpha_1 - v_{\alpha 1} \cos \alpha_1 + v_{\delta 1} \sin \delta_1 \sin \alpha_1 = -18.33$$

$$v_{z1} = v_{r1} \sin \delta_1 + v_{\delta 1} \cos \delta_1 = -0.56$$

Similarly, Cartesian velocity vector of Altair will be, $v_2 = (4.33, -33.85, 9.87)$. (8 points)

Lastly, we evaluate the value of determinant,

$$\begin{vmatrix} (x_1 - x_2) & (y_1 - y_2) & (z_1 - z_2) \\ v_{x1} & v_{y1} & v_{z1} \end{vmatrix} = \begin{vmatrix} -1.40 & 1.42 & 4.02 \\ 4.45 & -18.33 & -0.56 \\ 4.33 & -33.85 & 9.87 \end{vmatrix}$$

$$= 279.82 - 65.81 - 286.48 = -72.47$$

As the value of the determinant is non-zero, the direction vectors of these two stars do not cross each other. Hence no such point can exist. (4 points)

(T11) Thermal History of the Universe points)

(75

Based on Einstein's general relativity, Russian physicist Alexander Friedmann derived the Friedmann Equation by which the dynamics of a homogeneous and isotropic universe can be well described. The Friedmann Equation is usually written as follows:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_m + \rho_r) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}.$$

We define the Hubble parameter as $\frac{\dot{a}}{a}$, where a is the scale factor and $\dot{a}$ is the rate of change of scale factor with time. Thus, the Hubble parameter is a function of cosmic time. In the Friedmann Equation, ρ_m is the density of matter, including dark matter and baryons, ρ_r is the density of radiation, Λ is the cosmological constant, and k is the curvature of space. Subscript 0 indicates the value of a physical quantity at present day, e.g. H_0 is the present value Hubble parameter. Also, to avoid confusion with the reduced Hubble parameter, we use the reduced Planck Constant $\hbar = h/(2\pi)$ instead of the Planck constant h .

(a) (5 points) What are the dimensions of Hubble parameter? One can define a characteristic timescale for the expansion of the Universe (i.e. Hubble time t_H) using the Hubble parameter. Calculate the present-day Hubble time t_{H0} .

(b) (5 points) Let us define the critical density ρ_c as the matter density required to explain the expansion of a flat universe without any radiation or dark energy. Find an expression of the critical density, in terms H and G . Calculate the present critical density ρ_{c0} .

(c) (6 points) It is convenient to define all density parameters in a dimensionless manner like $\Omega_i = \frac{\rho_i}{\rho_c}$, i.e. the ratio of density to critical density. The Friedmann Equation can be rewritten using these dimensionless density parameters simply as, $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$.

Use this information to find expression for Ω_Λ and Ω_k , in terms H , c , Λ , k and a .

(d) (7 points) Another equation which is valid for matter, radiation and dark energy is often called the Fluid Equation: $\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0$, where p is the pressure of some component, ρ is the density and $\dot{\rho}$ is the rate of change of density over time. Radiation contains photons and massless neutrinos, and they both travel at the speed of light. The pressure exerted by these particles is 1/3 of their energy density. Show that the density of radiation $\rho_r \propto (1+z)^4$, where z is cosmological redshift. You may note that if $\frac{\dot{\rho}}{\rho} = n \frac{\dot{a}}{a}$, then $\rho \propto a^n$.

(e) (4 points) We know that the value of the cosmological constant Λ doesn't evolve. Its equation of state has a form $p = w\rho_\Lambda c^2$, where w is an integer. Find the value of w .

(f) (13 points) Planck time, defines a characteristic timescale before which our present physical laws are no longer valid, and where quantum gravity is needed. The expression for Planck time can be written in terms of $\hbar$, G and c and non-dimensional coefficient of this expression in SI units is of the order of unity. Using dimensional analysis, find expression for Planck time and estimate its value.

(g) (7 points) Planck length defines the length scale associated with Planck time is given by $l_P = ct_P$. The minimal mass of a black hole, also called Planck mass, is defined as the mass of a black hole whose Schwarzschild radius is two times the Planck length.

Derive the Planck mass M_P and calculate $M_P c^2$ in GeV. This mass is considered to be an upper threshold for elementary particles, beyond which they will collapse to a black hole.

(h) (4 points) At the very beginning (soon after the Planck time), all the particles were in thermal equilibrium in a primordial soup. As temperature decreased, different particles then decoupled from the primordial soup one by one and could travel freely in the Universe. Photons decoupled at ~ 300000 years after the Big Bang. These photons emitted at that time are what constitutes the cosmic microwave background (CMB), which follows the Stefan-Boltzmann law for blackbody radiation.

$$\epsilon_r = \frac{\pi^2}{15\hbar^3 c^3} (k_B T)^4,$$

Show that the temperature of the CMB follows $T/(1+z) = \text{constant}$.

(i) (16 points) With the expansion of the Universe, radiation density dropped more quickly than matter density, and at some epoch the matter density was equal to the radiation density. Radiation contains both photons and neutrinos. Apart from photons, neutrinos additionally contribute to the radiation energy density by 68% (i.e. $\Omega_{r0} = 1.68\Omega_{\gamma 0}$, where γ indicates photons). Estimate the redshift of matter-radiation equality z_{eq} in terms of Ω_{m0} and reduced Hubble parameter $h = \frac{H_0}{100 \text{ km s}^{-1} \text{ Mpc}^{-1}}$. You may use the current temperature of the CMB: $T_0 = 2.73 \text{ K}$.

(j) (8 points) The neutrinos decoupled from the primordial soup when the temperature of the universe was around 1 MeV. At this time, the radiation density in the universe was much more than all other components. Estimate the time ($t = \frac{1}{2H}$) when neutrinos decoupled, and express it in seconds since the big bang.

Solution:

(a) One can argue this in different ways. One can note that the Hubble parameter is often called Hubble constant, but actually it's not a constant. The dimensions of Hubble parameter is the inverse of time $[T^{-1}]$. Alternatively, one can simply look at unit of H_0 in the table of constants and conclude the same. (1 point)

It's natural to define a timescale as the reciprocal of Hubble parameter: $t_H = \frac{1}{H}$, and this is the Hubble time which is a characteristic timescale of Universe expansion.

(2 points)

Present-day Hubble time $t_{H0} = 14.46 \text{ Gyr}$. (2 points)

(b) From Friedmann Equation, the critical density is defined in the way of $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_c$, thus $\rho_c = \frac{3H^2}{8\pi G}$.

(2 points)

Substitute $H_0 = 2.19 \times 10^{-18} \text{ s}^{-1}$ into above, we have $\rho_{c0} = 8.59 \times 10^{-27} \text{ kg/m}^3$ (3 points)

(c) $\Omega_m + \Omega_r + \Omega_\Lambda + \Omega_k = 1$

$$\frac{\rho_m}{\rho_c} + \frac{\rho_r}{\rho_c} + \Omega_\Lambda + \Omega_k = 1$$

$$\frac{8\pi G}{3H^2} (\rho_m + \rho_r) + \Omega_\Lambda + \Omega_k = 1$$

$$\frac{8\pi G}{3} (\rho_m + \rho_r) + H^2 \Omega_\Lambda + H^2 \Omega_k = H^2 \quad (2 \text{ points})$$

$$\text{Comparing this with the Friedmann equation } H^2 = \frac{8\pi G}{3} (\rho_m + \rho_r) + \frac{\Lambda c^2}{3} - \frac{kc^2}{a^2}.$$

$$\Omega_\Lambda = \frac{\Lambda c^2}{3H^2} \text{ and } \Omega_k = -\frac{kc^2}{a^2 H^2} \quad (4 \text{ points})$$

(d) Pressure exerted by the radiation is $p = \frac{1}{3}\rho_r c^2$

Thus, the Fluid Equation becomes, $\dot{\rho}_r + 3\frac{\dot{a}}{a}\left(\rho_r + \frac{1}{3}\rho_r\right) = 0$ (2 points)

$$\frac{\dot{\rho}_r}{\rho_r} = -4\frac{\dot{a}}{a} \text{ implying}$$

Thus $\rho_r \propto a^{-4}$. (3 points)

And we know $a = \frac{1}{1+z}$, hence $\rho_r \propto (1+z)^4$. (2 points)

(e) In the fluid equation, $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + q\rho) = 0$.

But the cosmological constant does not evolve. So $\dot{\rho}_\Lambda = 0$

$$\frac{3\dot{a}\rho_\Lambda}{a}(1+q) = 0 \text{ which means } q = -1. \quad (4 \text{ points})$$

(f) From the units, one can figure out that $\hbar$ has the dimensions of $[ML^2T^{-1}]$.

G has the dimensions of $[M^{-1}L^3T^{-2}]$.

And speed of light c has dimensions of $[LT^{-1}]$. (5 points)

We assume that Planck time is given by $\hbar^x c^y G^z$. In order to make it a time dimensional quantity, the following equation must hold: $T = (ML^2T^{-2})^x (LT^{-1})^y (M^{-1}L^3T^{-2})^z$. $T = M^{x-z} L^{2x+y+3z} T^{-x-y-2z}$.

(3 points)

From $x - z = 0, 2x + y + 3z = 0, -x - y - 2z = 1$, we get $x = \frac{1}{2}, y = \frac{-5}{2}, z = \frac{1}{2}$. Thus, the Planck time is approximately $t_p \approx \sqrt{\hbar G/c^5}$. (3 points)

points)

Substitute the numerical values into it and get: $t_p = 5.4 \times 10^{-44} s$. (2 points)

(g) Schwarzschild radius of a black hole equals to $2l_P$: $r_s = \frac{2GM_P}{c^2} = 2ct_p = 2c\sqrt{\hbar G/c^5}$, (3 points)

$$M_P = \sqrt{\hbar c/G}. \quad (2 \text{ points})$$

$$M_P c^2 = \sqrt{\hbar c^5/G} = 1.22 \times 10^{19} \text{ GeV}. \quad (2 \text{ points})$$

(h) Given the energy density, and the result of problem d, we find that $\varepsilon_r \propto T^4, \varepsilon_r = \rho_r c^2, \rho_r \propto (1+z)^4$.

Thus $T \propto 1+z, \frac{T}{1+z} = \text{const.}$ (4 points)

(i) The matter-radiation equality means $\Omega_r(z_{eq}) = \Omega_m(z_{eq})$. We need to derive the behavior of this two density parameters. Obviously $\Omega_m(z) = \Omega_{m0}(1+z)^3$ and $\Omega_r(z) = \Omega_{r0}(1+z)^4$. (4 points)

Thus $1+z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}}$. Ω_{r0} is the density parameter of CMB radiation and can be calculated. (1 point)

By the definition of density parameter of photon radiation (footnote gamma: $\gamma, \Omega_{\gamma 0} = \frac{\rho_{\gamma 0}}{\rho_{c0}} = \frac{\varepsilon_{\gamma 0}}{c^2} \frac{8\pi G}{3H^2} =$

$$\frac{\varepsilon_{\gamma 0}}{c^2} \frac{8\pi G}{3H_{100}^2 h^2}, \text{ where } H_{100} = 100 \text{ km/s/Mpc}. \quad (4 \text{ points})$$

The energy density of blackbody photon radiation is $\varepsilon_\gamma = \frac{\pi^2}{15\hbar^3 c^3} (k_B T)^4$, thus

$$\Omega_{\gamma 0} h^2 = \frac{\pi^2}{15\hbar^3 c^5} k_B^4 \frac{8\pi G}{3H_{100}^2} T_0^4 = 2.47 \times 10^{-5}. \quad (4 \text{ points})$$

But there is also an additional contribution from neutrinos of about 68%, so the total density parameter of radiation is $\Omega_{r0} h^2 = \Omega_{\gamma 0} h^2 \times 1.68 = 4.15 \times 10^{-5}$. (2 points)

Hence $1 + z_{eq} = \frac{\Omega_{m0}}{\Omega_{r0}} = 2.4 \times 10^4 \Omega_{m0} h^2$. (1 point)

(j) At that time, the redshift is

$$1 + z = \frac{1 \text{ MeV}}{k_B \times 2.73 \text{ K}} = 4.25 \times 10^9.$$

Thus the scale factor $a = \frac{1}{1+z} = 2.35 \times 10^{-10}$. (2 points)

Only considering radiation dominated Universe, Friedmann Equation can be written as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3} \rho_r = \frac{8\pi G \rho_{r0}}{3a^4}, \quad (2 \text{ points})$$

$$\frac{\dot{a}}{a} = \frac{1}{a^2} \sqrt{\frac{8\pi G \rho_{r0}}{3}}$$

$$a^2 H = \sqrt{\frac{8\pi G \Omega_{r0}}{3}} \rho_{c0} = \sqrt{H_0^2 \Omega_{r0}} \quad (1 \text{ points})$$

Thus, we have, $t = \frac{1}{2H} = \frac{a^2}{2\sqrt{H_0^2 \Omega_{r0}}} = 1.3 \text{ s} \sim 1 \text{ s}.$

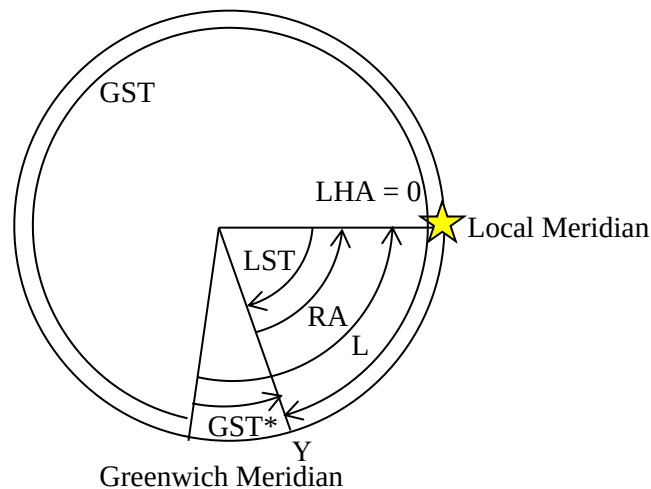
Thus neutrino decoupled at about 1 s after big bang. (3 points)

(T1) The Large Magellanic Cloud in Phuket

[10 marks]

The coordinates of the Large Magellanic Cloud (LMC) are R.A. = 5h 24min and Dec = $-70^{\circ}00'$. The latitude and longitude of Phuket are $7^{\circ}53' N$ and $98^{\circ}24' E$, respectively. What is the date when the LMC culminates at 9pm as seen from Phuket in the same year? You may note that the Greenwich sidereal time, GST, at 00h UT 1st January is about 6h 43min, and Phuket follows UT+7 time zone.

Solution:



When it culminates, $LHA = 0$, $RA = -LST$ [1.0]

$$LST = -(L - GST^*) = -RA$$

$$GST^* = L - RA = 6h\ 34min - 5h\ 24min = 1h\ 10min$$
 [2.0]

$$GST = 24h - 1h\ 10min$$

$$GST = 22h\ 50min$$
 [1.0]

Phuket time of 21:00 is UT14:00. [1.0]

$$GST = GST\ of\ 1^{st}\ Jan + \text{The angle that } \Upsilon \text{ moves away from } 1^{st}\ Jan + UT$$
 [1.0]

$$22h\ 50min = 6h\ 43min + \left[\left(\frac{\text{Days that away from } 1^{st}\ Jan}{365.2422\ days} \right) \times 24\ h \right] + \left[14h \times \left(\frac{24}{23.9344} \right) \right]$$

$$\text{Days that away from } 1^{st}\ Jan = (22h\ 50min - 6h\ 43min - 14h\ 2min) \left(\frac{365.2422\ days}{24h} \right)$$
 [2.0]

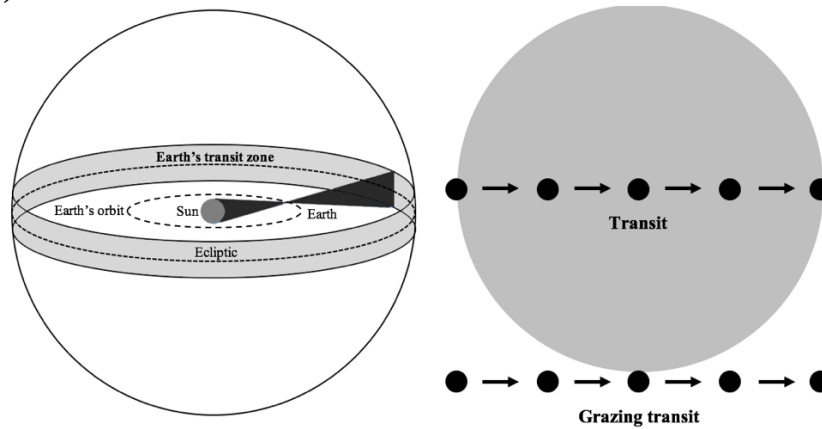
$$\text{Days that away from } 1^{st}\ Jan = 31.7\ days\ from\ 1^{st}\ Jan\ (at\ Greenwich)$$
 [1.0]

The time of the year that the LMC culminates in Phuket is on 2nd February. [1.0]

(T2) Earth's transit zone

[10 marks]

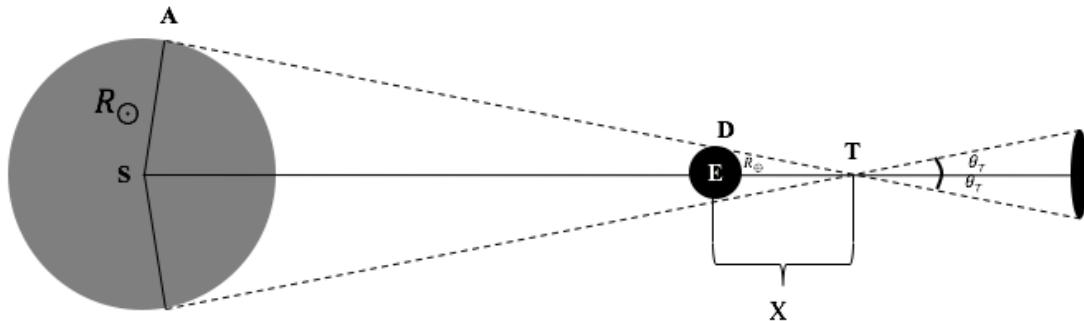
Earth's transit zone is an area where extrasolar observers (located far away from the Solar system) can detect the Earth transiting across the Sun. For observers on the Earth, this area is the projection of a band around the Earth's ecliptic onto the celestial plane (light grey area in the left figure). Assume that the Earth has a circular orbit of 1 au.



- a) Find the angular width of that part of the Earth's transit zone in degrees, where the extrasolar observers can detect Earth's total transit (the whole Earth's disk passing in front of the Sun). [5]

Solution:

For Earth's transit, the whole Earth's disk should pass in front of the Sun



From $\Delta STA \sim \Delta ETD$:

[1.0]

$$\frac{X}{R_{\oplus}} = \frac{a + X}{R_{\square}}$$

$$X = \frac{aR_{\oplus}}{R_{\square} - R_{\oplus}}$$

[1.0]

The half angular size of the Earth's Transit Zone with transit can be written as,

$$\theta_T = \arcsin\left(\frac{R_{\oplus}}{X}\right) = \arcsin\left(\frac{R_{\square}}{a + X}\right) = \arcsin\left(\frac{R_{\square} - R_{\oplus}}{a}\right)$$

[1.0]

Solution with $\theta_T = \arctan\left(\frac{R_{\square} - R_{\oplus}}{a}\right) \approx \frac{R_{\square} - R_{\oplus}}{a}$ is acceptable with full mark

The angular size of the Earth's Transit Zone with transit is

$$2\theta_T = 2 \arcsin \left(\frac{R_{\square} - R_{\oplus}}{a} \right) \quad [1.0]$$

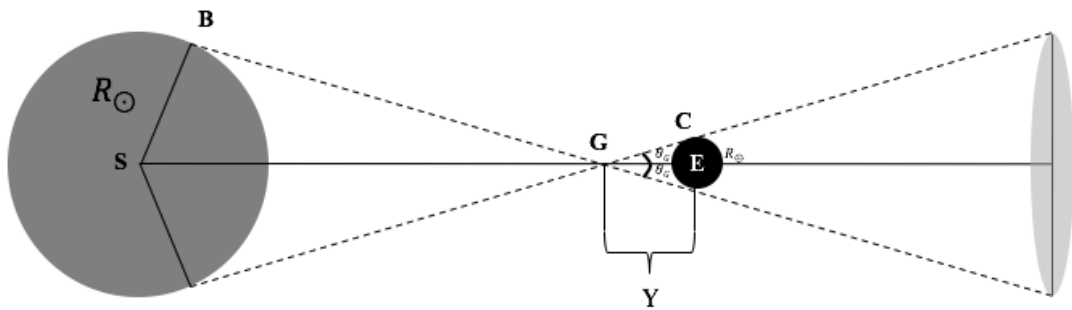
$$2\theta_T = 0.527^{\circ} \quad [1.0]$$

- b) Find the angular width of that part of Earth's transit zone in degrees, where the extrasolar observers can detect at least Earth's grazing transit (any part of the Earth's disk passing in front of the Sun).

[5]

Solution:

For Earth's grazing transit, any part of the Earth's disk should pass in front of the Sun



From $\triangle SGB \sim \triangle EGC$: [1.0]

$$\frac{Y}{R_{\oplus}} = \frac{a - Y}{R_{\square}}$$

$$Y = \frac{aR_{\oplus}}{R_{\square} + R_{\oplus}} \quad [1.0]$$

The angular size of the Earth's Transit Zone with transit can be written as,

$$\theta_G = \arcsin \left(\frac{R_{\oplus}}{Y} \right) = \arcsin \left(\frac{R_{\square}}{a - Y} \right) = \arcsin \left(\frac{R_{\square} + R_{\oplus}}{a} \right) \quad [1.0]$$

Solution with $\theta_G = \arctan \left(\frac{R_{\square} + R_{\oplus}}{a} \right) \approx \frac{R_{\square} + R_{\oplus}}{a}$ is acceptable with full mark

The angular size of the Earth's Transit Zone with grazing transit is

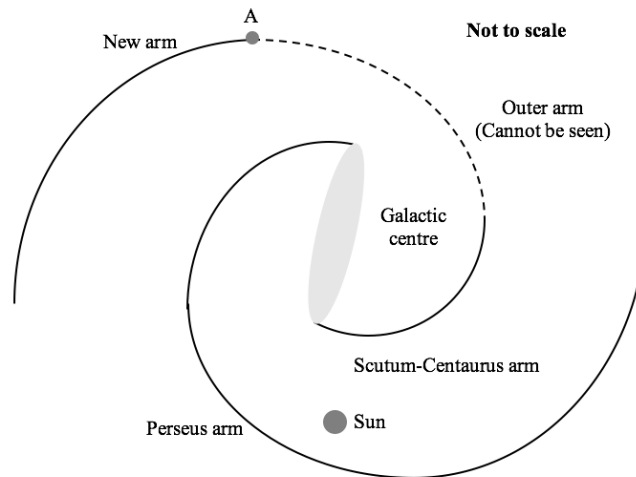
$$2\theta_G = 2 \arcsin \left(\frac{R_{\square} + R_{\oplus}}{a} \right) \quad [1.0]$$

$$2\theta_G = 0.537^{\circ} \quad [1.0]$$

(T3) Milky Way New Far Outer Arm

[10 marks]

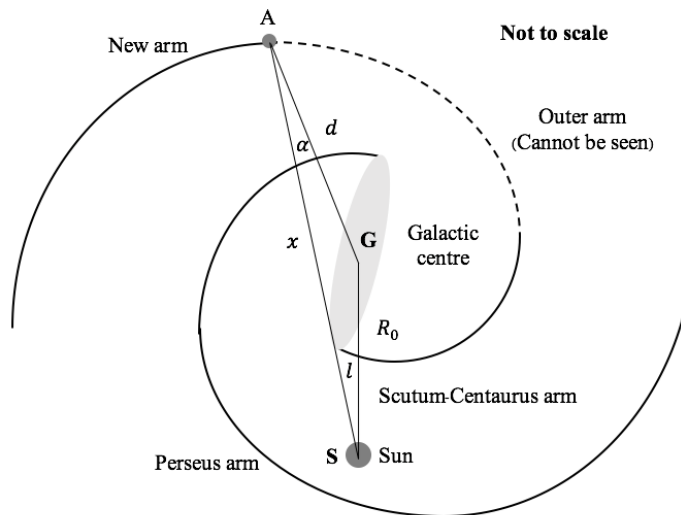
In 2011, Dame and Thaddeus found a new part of outer arm of the Milky Way by studying the CO line using the CfA 1.2m telescope. They detected that the distribution of CO starts at galactic longitude $\ell = 13.25^\circ$ (marked **A** in the figure) where it has radial velocity of 20.9 km s^{-1} towards the Sun. Assume that the galactic rotation curve is flat beyond 5 kpc from the galactic centre. The distance between the Sun and the Galactic centre is 8.5 kpc. The velocity of the Sun around the Galactic centre is 220 km s^{-1} .



a) Find the distance from the start of the arm (point **A**) to the Galactic centre.

[7]

Solution:



If there is the flat rotation curve beyond 2 kpc from the Galactic centre, the orbital velocity of the Sun and the start of the arm are both equal to

$$v_r = -v_{\text{LSR}} \sin(\ell) + v_{\text{LSR}} \sin(\alpha)$$

[3.0]

$$\text{Sine law (} \Delta \text{AGS): } \frac{\sin(\ell)}{d} = \frac{\sin(\alpha)}{R_0} \quad [1.0]$$

$$v_r = -v_{\text{LSR}} \sin(\ell) + v_{\text{LSR}} R_0 \frac{\sin(\ell)}{d} \quad [1.0]$$

$$20.9 = 220 \sin(13.25^\circ) - 220 \times 8.5 \times \frac{\sin(13.25^\circ)}{d} \quad [1.0]$$

Correctly substitute numerical values

$$d = 14.5 \text{ kpc} \quad [1.0]$$

b) Find the distance from the start of the arm (point **A**) to the Sun. [3]

Solution:

$$\text{Cosine law (} \Delta \text{AGS): } d^2 = x^2 + R_0^2 - 2xR_0 \cos(\ell) \quad [1.0]$$

$$14.5^2 = x^2 + 8.5^2 - 2 \times 8.5x \cos(13.25^\circ) \quad [1.0]$$

Correctly substitute numerical values

$$x = \frac{17 \cos(13.25^\circ) + \sqrt{17^2 \cos^2(13.25^\circ) - 4 \times (8.5^2 - 14.5^2)}}{2}$$

$$x = 22.6 \text{ kpc} \quad [1.0]$$

(T4) 21-cm HI galaxy survey

[10 marks]

A radio telescope is equipped with a receiver which can observe in a frequency range from 1.32 to 1.52 GHz. Its detection limit is 0.5 mJy per beam for a 1-minute integration time. In a galaxy survey, the luminosity of the HI spectral line of a typical target galaxy is 10^{28} W with a linewidth of 1 MHz. For a large beam, the HI emitting region from a far-away galaxy can be approximated as a point source. The HI spin-flip spectral line has a rest-frame frequency of 1.42 GHz.

What is the highest redshift z of a typical HI galaxy that can be detected by a survey carried out with this radio telescope, using a 1-minute integration time? You may assume in your calculation that the redshift is small and the non-relativistic approximation can be used. Note that $1 \text{ Jy} = 10^{-26} \text{ Wm}^{-2}\text{Hz}^{-1}$.

Solution:

Case 1: If we only consider the frequency range where HI spectral line can be detected by the receiver of this radio telescope,

$$z \approx \frac{f_0 - f}{f_0} \quad [1 \text{ Mark}]$$

The lowest frequency, 1.32 GHz, can be used to observe the highest redshifted HI, at

$$z = \frac{1.42 - 1.32}{1.42} = 0.0704 \quad [1 \text{ Mark}]$$

Alternative solution: Above is the cosmological redshift formula in frequency conventionally used in radio astronomy. However, if the cosmological redshift in optical astronomy is used, student may use

$$z \approx \frac{f_0 - f}{f} = \frac{1.42 - 1.32}{1.32} = 0.0758$$

and the student should be awarded full mark for this part.

Case 2: Use the information given to calculate the redshift limit set by the flux limit due to furthest distance that typical HI galaxy can be detected by the telescope

$$\text{for low } z, \text{ non-relativistic redshift } d = \frac{v_r}{H_0} = \frac{cz}{H_0} \quad [1 \text{ Mark}]$$

The HI spectral-line flux density for a galaxy with HI luminosity L and line-width Δf at distance d is

$$\frac{L}{4\pi d^2} \times \frac{1}{\Delta f} (\text{Wm}^{-2}\text{Hz}^{-1}) \quad [2 \text{ Mark}]$$

Setting this to the detection limit and solve correctly, and taking special care of converting various units

$$S = \frac{LH_0^2}{4\pi\Delta f c^2 z^2} \geq S_{\text{lim}} = 0.5 \times 10^{-26} \quad [1 \text{ Mark}]$$

Therefore,

$$z \leq \frac{H_0}{c} \sqrt{\frac{L}{4\pi\Delta f S_{\text{lim}}}}, \quad [1 \text{ Mark}]$$

Correctly substitute numerical values,

$$z \leq \frac{67.8(\text{km}^{-1}\text{Mpc}^{-1})}{2.99 \times 10^5 (\text{km s}^{-1}) \times 3.08 \times 10^{22} (\text{m Mpc}^{-1})} \sqrt{\frac{10^{28} \text{ W}}{4\pi \times 10^6 (\text{Hz}) \times 0.5 \times 10^{-29} (\text{Wm}^{-2}\text{Hz}^{-1})}} \quad [1 \text{ Mark}]$$

$$z \leq 0.0929 \quad [1 \text{ Mark}]$$

Conclusion: For typical HI galaxy, the highest redshift is limited by the receiver's lowest frequency limit and we get $z_{\text{max}}=0.0704$ (or 0.0758 for alternative solution given above) [1 Marks]

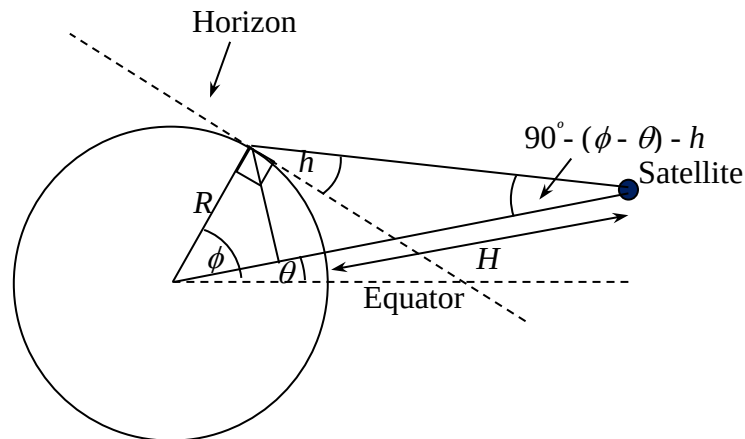
*The last part is marked only if student calculate both instrumental factors.
If students use more precise cosmological effects, they will be marked accordingly.*

(T5) A Synchronous Satellite

[10 marks]

A synchronous satellite is a satellite which orbits the Earth with its period exactly equal to the period of rotation of the Earth. The height of these satellites is 35,786 km above the surface of the Earth. A satellite is put at an inclined synchronous orbit with the inclination $\theta = 6.69^\circ$ to the equatorial plane. Calculate the precise value of the possible maximum altitude of the satellite for an observer at latitude of $\phi = 51.49^\circ$.

Solution:



Draw diagram and correctly identify the altitude [1.0]

Identify another angle with the altitude h in its expression, such as $90^\circ - (\phi - \theta) - h$ [1.0]

Maximum altitude occurs when the satellite is at meridian [1.0]

Intermediate steps that can lead to the calculation of h , such as
$$\tan(90^\circ - (\phi - \theta) - h) = \frac{R \sin(\phi - \theta)}{H + R - R \cos(\phi - \theta)}$$
 [3.0]

Correct expression for h : [2.0]

$$h = 90^\circ - (\phi - \theta) - \tan^{-1} \left(\frac{R \sin(\phi - \theta)}{H + R - R \cos(\phi - \theta)} \right)$$

Correct value of h

$$h = 90^\circ - (51.49^\circ - 6.69^\circ) - \tan^{-1} \left(\frac{\sin(51.49^\circ - 6.69^\circ)}{\frac{35,786 \text{ km}}{6.38 \times 10^3 \text{ km}} + 1 - \cos(51.49^\circ - 6.69^\circ)} \right)$$

$h = 38.4^\circ$ [2.0]

Maximum of 2.0 marks for solution with $h = 90.0^\circ - 51.49^\circ + 6.69^\circ = 45.2^\circ$

(T6) Supernova 1987A

[15 marks]

Supernova SN 1987A was at its brightest with apparent magnitude of +3 on about 15th May 1987 and then faded, finally becoming invisible to the naked eye by 4th February 1988. It is assumed that brightness B varied with time t as an exponential decline, $B = B_0 e^{-t/\tau}$, where B_0 and τ are constant. The maximum apparent magnitude which can be seen by the naked eye is +6.

a) Determine the value of τ in days.

[5]

Solution:

$$m - m_0 = -2.5 \log_{10} \left(\frac{B}{B_0} \right) \quad [1.0]$$

$$m - m_0 = 2.5 \frac{t}{\tau} \log_{10} e \quad [1.0]$$

The days from 15th May 1987 to 4th February 1988 is 265 days [1.0]

$$6 - 3 = 2.5 \frac{265 \text{ days}}{\tau} \log_{10} e \quad [1.0]$$

$$\tau = 95.9 \text{ days} \quad [1.0]$$

b) Find the last day that observers could have seen the supernova if they had a 6 inch (15.24 cm) telescope with transmission efficiency $T = 70\%$. Assume that the average diameter of the human pupil is 0.6 cm. [10]

Solution:

Consider the energy transmission

$$B_e d^2 = T B_T D^2 \quad [3.0]$$

$$\frac{B_e}{B_T} = T \frac{D^2}{d^2}$$

$$m - m_0 = -2.5 \log_{10} \left(\frac{B}{B_0} \right) \quad [1.0]$$

$$m_{\text{lim}} = m_e + 2.5 \log_{10} T + 5 \log_{10} \left(\frac{D}{d} \right) \quad [1.0]$$

$$m_{\text{lim}} = 6 + 2.5 \log_{10} 0.7 + 5 \log_{10} \left(\frac{15.24 \text{ cm}}{0.6 \text{ cm}} \right)$$

$$m_{\text{lim}} = 12.64$$

and

$$\text{From } m - m_0 = 2.5 \frac{t}{\tau} \log_{10} e \quad [2.0]$$

$$12.64 - 3 = 2.5 \frac{t}{95.9 \text{ days}} \log_{10} e \quad [1.0]$$

$$t = 851.5 \text{ days} \quad [1.0]$$

The last day that observers could have seen the supernova is on **12th September 1989**

[1.0]

11th - 13th September 1989 are acceptable

(T7) Life on Other Planets

[20 marks]

One place to search for life is on planets orbiting main sequence stars. A good starting point is the planets that have an Earth-like temperature range and a small temperature fluctuation. Assume that for a main sequence star, the relation between the luminosity L and the mass M is given by

$$L \propto M^{3.5}.$$

You may assume that the total energy E released over the lifetime of the star is proportional to the mass M of the star. For the Sun, it will have a main sequence lifetime of about 10 billion years. The stellar spectral types are given in the table below. Assume that the spectral subclasses of stars (0-9) are assigned on a scale that is linear in $\log M$.

Spectral Class	O5V	B0V	A0V	F0V	G0V	K0V	M0V
Mass ($M_{\odot}$)	60	17.5	2.9	1.6	1.05	0.79	0.51

- a) If it takes at least 4×10^9 years for an intelligent life form to evolve, what is the spectral type (accurate to the subclass level) of the most massive star in the main sequence around which astronomers should look for intelligent life? [6]

Solution:

Since $E \propto M \rightarrow L\tau \propto M$ and $L \propto M^{3.5}$,

$$\tau = \tau_{\odot} \left(\frac{M_{\odot}}{M} \right)^{2.5} \quad [2.0]$$

The maximum mass of the star should be

$$M = M_{\odot} \left(\frac{\tau_{\odot}}{\tau} \right)^{1/2.5} = M_{\odot} \left(\frac{10^{10}}{4 \times 10^9} \right)^{1/2.5} \quad [0.5]$$

$$M = 1.44 M_{\odot} \quad [0.5]$$

Use of spectral class-log M linear relation

$$\frac{10}{(\log 1.6 - \log 1.05)} \times (\log 1.6 - \log 1.44) = 2.5 \quad [2.0]$$

which corresponds to the star of type **F2V** to **F3V** [1.0]

0.5 mark for the answer "between F0V and G0V"

- b) Assume that the target planet has the same emissivity ε and albedo a as the Earth. In order to have the same temperature as the Earth, express the distance d , in au, of the planet to its parent main sequence star, of mass M . [6]

Solution:

The received power by a planet of a radius r and a distance d from the star is

$$P_{input} = (1-a) \frac{\pi r^2 L}{4\pi d^2} = (1-a) \frac{r^2 L}{4d^2}, \text{ where } a \text{ is the planet's albedo. [1.0]}$$

The released power of blackbody radiation is

$$P_{output} = 4\pi r^2 \varepsilon \sigma T^4, \text{ where } \varepsilon \text{ is emissivity. [1.0]}$$

Since we assume that the planet emits heat as blackbody radiation,

$$(1-a) \frac{r^2 L}{4d^2} = 4\pi r^2 \varepsilon \sigma T^4. [1.0]$$

The temperatures of the planet and the Earth are similar,

$$T^4 = \frac{(1-a)L}{16\pi\varepsilon\sigma d^2} = \frac{(1-a)L_{\odot}}{16\pi\varepsilon\sigma d_{\odot}^2}.$$

$$\frac{L}{d^2} = \frac{L_{\odot}}{d_{\odot}^2} [1.0]$$

$$\left(\frac{d}{d_{\odot}}\right)^2 = \frac{L}{L_{\odot}} = \left(\frac{M}{M_{\odot}}\right)^{3.5} [1.0]$$

The distance of the planet to the star is $d = \left(\frac{M}{M_{\odot}}\right)^{3.5/2}$ au. [1.0]

- c) The existence of a planet around a star can be shown by the variation in the radial velocity of the star about the star-planet system centre of mass. If the smallest Doppler shift in the wavelength detectable by the observer is $(\Delta\lambda / \lambda) = 10^{-10}$, calculate the lowest mass of such a planet in b), in units of Earth masses, that can be detected by this method, around the main sequence star in a). [8]

Solution:

Angular speed around the center of mass can be derived by Kepler's law.

$$T^2 = \frac{4\pi^2 d^3}{G(M+m)} \quad [1.0]$$

$$\omega = \sqrt{\frac{G(M+m)}{d^3}} \quad [1.0]$$

According to the centre of mass of the planet and the star,

$$\begin{aligned} Md_s &= md \\ d_s &= \frac{m}{M} d \end{aligned} \quad [1.0]$$

and

$$v = \omega d_s \quad [1.0]$$

$$v = \omega d \frac{m}{M} = \frac{m}{M} \sqrt{\frac{G(M+m)}{d}} \approx m \sqrt{\frac{G}{Md}}$$

From $\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = \frac{m}{c} \sqrt{\frac{G}{Md}} \quad [1.0]$

Use of previously derived expression The distance of the planet to the star is

$$d = \left(\frac{M}{M_\odot} \right)^{3.5/2} \text{ au.} \quad [1.0]$$

$$\begin{aligned} m &= \frac{\Delta\lambda}{\lambda} c \sqrt{\frac{Md}{G}} = \frac{\Delta\lambda}{\lambda} c \sqrt{\frac{M \left(\frac{M}{M_\odot} \right)^{3.5/2} \times 1 \text{ au}}{G}} \\ &= \frac{\Delta\lambda}{\lambda} c \sqrt{\frac{M (1.44)^{3.5/2} \times 1 \text{ au}}{G}} \\ &= (299792458 \text{ m/s}) \times 10^{-10} \sqrt{\frac{1.44 \times 1.99 \times 10^{30} \text{ kg} (1.44)^{3.5/2} \times 1.50 \times 10^{11} \text{ m}}{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}}} \end{aligned} \quad [1.0]$$

$$m = 3.31 \times 10^{24} \text{ kg} = 0.554 M_\oplus \quad [1.0]$$

(T8) Star of Bethlehem

[20 marks]

A great conjunction is a conjunction of Jupiter and Saturn for observers on Earth. Assume that Jupiter and Saturn have circular orbits in the ecliptic plane.

The time between successive conjunctions may vary slightly as viewed from the Earth. However, the average time period of the great conjunctions is the same as that of an observer at the centre of the Solar system.

- a. Find the average great conjunction period (in years) and average heliocentric angle between two successive great conjunctions (in degrees). [6]

Solution:

Assume that the average Earth's orbit is at the Sun. Therefore, average great conjunction period equals to the synodic period between Jupiter and Saturn.

$$\frac{1}{P_{\text{Synodic}}} = \frac{1}{P_{\text{Jupiter}}} - \frac{1}{P_{\text{Saturn}}} \quad [3.0]$$

$$\frac{1}{P_{\text{Synodic}}} = \frac{1}{11.86} - \frac{1}{29.45}$$

$$P_{\text{Synodic}} = 19.86 \text{ years} \quad [1.0]$$

From average great conjunction period, in 19.86 years Jupiter and Saturn move

$$\text{Heliocentric angle} = \frac{19.86}{11.86} \times 360^\circ = 602.7^\circ = 242.7^\circ$$

in an eastward direction through the zodiac, or $360^\circ - 242.7^\circ = 117.3^\circ$ in a westward direction through the zodiac. [2.0]

- b. The next great conjunction will be on 21st December 2020 with an elongation of 30.3° East of the Sun. Estimate in which constellation will the conjunction on 21st December 2020 occur? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [2]

Solution:

On 21st December 2020 (winter solstice), the Sun will be in the constellation of Sagittarius. Planets are always near the ecliptic. The current 12 zodiac constellations can be roughly divided into 12 equal parts of 30 degrees each.

The conjunction will locate elongation 30.3° East to the Sun. Therefore, the conjunction will be in the constellation of **Capricornus (Cap)**. [2.0]

Justification is NOT necessary for the full mark for this part

In 1606, Johannes Kepler determined that in some years the great conjunction can be happen thrice in the year due to the retrograde motions of the planets. He also determined that such an event happened in the year 7BC, which could have been the event commonly known as “The Star of Bethlehem”. For the calculations below you may ignore the precession of the axis of the Earth.

- c. Estimate in which constellation did the great conjunctions in 7 BC occur? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [8]

Solution:

The duration between 7BC and 2020 is 2026 years or 2027 years .

Each conjunction occurs on the average every 19.86 years. Therefore, the number of conjunction between 7BC and 2020 is

Integer value of $2026 / 19.86 = 102$ [2.0]

Correct number of conjunctions 1.0 mark.

Correct number of conjunctions with integer 2.0 marks.

From a conjunction to its previous successive conjunction, the conjunction moves 243° westwards. The position of the conjunction in 7BC is

$102 \times 242.7^\circ = 24758.33^\circ = 278.3^\circ$ [3.0]

westward, or equivalently 81.7° eastward from the position of the conjunction in 2020. Therefore, the conjunction was in the constellation of **Aries (Ari)** or **Pisces (Psc)** [3.0]

Justification is NOT necessary for the full mark for this part
Correct answer only

- d. At the second conjunction of the series of three conjunctions in 7 BC, for the observer on Earth, estimate in which constellation was the Sun? (Give the IAU Latin name or IAU three-letter abbreviation of the constellation, i.e. Ursa Major or UMa) [4]

Solution:

At the second conjunction, Jupiter and Saturn had retrograde motions. For observers on the Earth, both of them were in opposition. Therefore, the sun was in the constellation of **Libra (Lib)** (if answer c) is Aries) or **Virgo (Vir)** (if answer c) is Pisces). [4.0]

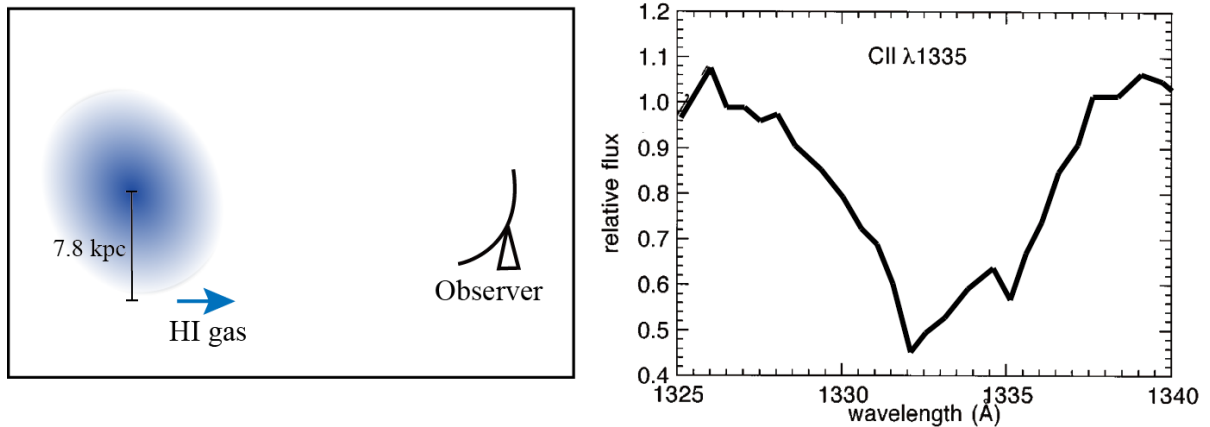
This question is marked only if students get c) correctly

(T2) Galactic Outflow

[20 marks]

Cannon et al. (2004) conducted an HI observation of a disk starburst galaxy, IRAS 0833+6517, with the Very Large Array (VLA). The galaxy is located at a distance of 80.2 Mpc with an approximate inclination angle of 23 degrees. According to the HI velocity map, IRAS 0833+6517 appears to be undergoing regular rotation with the observed radial velocity of the HI gas of roughly 5850 km s^{-1} at a distance of 7.8 kpc from the centre (the left panel of the figure below).

Gas outflow from IRAS 0833+6517 is traced by using the blueshifted interstellar absorption lines observed against the backlight of the stellar continuum (the right panel of the figure). Assuming that this galaxy is gravitationally stable and all the stars are moving in circular orbits,



- a) Determine the rotational velocity (v_{rot}) of IRAS 0833+6517 at the observed radius of HI gas. [5]

Solution:

The systemic or recessional velocity of IRAS 0833+6517 can be determined by the Hubble's law

$$\begin{aligned} v_{\text{sys}} &= H_0 d \\ &= 67.8 \text{ km s}^{-1} \text{Mpc}^{-1} \times 80.2 \text{ Mpc} \\ &= 5437.6 \text{ km s}^{-1} \end{aligned} \quad [2]$$

The observed radial velocity can be written in terms of systemic velocity and rotational velocity of a galaxy as

$$v_{\text{rad}} = v_{\text{sys}} + v_{\text{rot}} \sin i, \quad [1]$$

where v_{rad} , v_{sys} , and v_{rot} are the radial, systemic, and rotational velocity of the galaxy and i is an inclination angle.

The rotational velocity of the galaxy can thus be obtained by

$$v_{\text{rot}} = \frac{v_{\text{rad}} - v_{\text{sys}}}{\sin i} = \frac{5850 - 5437.6 \text{ km s}^{-1}}{0.39} = 1055 \text{ km s}^{-1} \quad [2]$$

Answers between 1050 to 1060 are acceptable for full marks.

- b) Calculate the escape velocity for a test particle in the gas outflow at the radius of 7.8 kpc. [9]

Solution:

Method 1:

In order to determine the escape velocity, we need to know the dynamical mass of the system. In case of the rotation dominated galaxy, a dynamical mass can be estimated as

$$M_{\text{dyn}}(< R) = v_{\text{rot}}^2 R / G, \quad [2]$$

where $M_{\text{dyn}}(< R)$ is the dynamical mass within the radius R .

So the dynamical mass of IRAS 0833+6517 within the radius of 7.8 kpc is

$$M_{\text{dyn}}(< R) = \frac{(1055 \times 10^3 \text{ m s}^{-1})^2 \times 7.8 \times 10^3 \text{ pc} \times 206265 \text{ au/pc} \times 1.50 \times 10^{11} \text{ m/au}}{6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}} = 4.0 \times 10^{42} \text{ kg} \quad [2]$$

$$\text{or } M_{\text{dyn}}(< R) = 2.0 \times 10^{12} M_{\odot}.$$

Assuming the virial theorem, the escape velocity can be written as

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \quad [2]$$

Therefore, the escape velocity is

$$v_{\text{esc}} = \sqrt{\frac{2 \times 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2} \times 4.0 \times 10^{42} \text{ kg}}{7.8 \times 10^3 \text{ pc} \times 206265 \text{ au/pc} \times 1.50 \times 10^{11} \text{ m/au}}} = 1492 \text{ km s}^{-1} \quad [3]$$

Method 2:

In order to determine the escape velocity, we need to know the dynamical mass of the system. In case of the rotation dominated galaxy, a dynamical mass can be estimated as

$$M_{\text{dyn}}(< R) = v_{\text{rot}}^2 R / G, \quad [2]$$

where $M_{\text{dyn}}(< R)$ is the dynamical mass within the radius R . According to the virial theorem, the escape velocity is

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} \quad [2]$$

Substituting the dynamical mass into the above equation, we get

$$v_{\text{esc}} = \sqrt{\frac{2GM}{R}} = \sqrt{2v_{\text{rot}}^2} \quad [3]$$

$$\begin{aligned} &= \sqrt{2 \times (1055 \text{ km s}^{-1})^2} \\ &= 1492 \text{ km s}^{-1} \end{aligned}$$

[2]

Answers between 1487 to 1497 are acceptable for full marks.

- c) Examine if the outflowing gas can escape from the galaxy at this radius by considering the velocity offset of the C II $\lambda 1335$ absorption line, which is already corrected for the recessional velocity. (The central wavelength of the CII absorption line in labs is 1335 Å.) [6]

Solution:

To examine whether or not outflowing gas can escape from the galaxy, one needs to derive the outflow velocity of the gas from the blueshifted absorption line.

The outflow velocity or the velocity offset of any emission or absorption lines can be calculated by the following equation:

$$\frac{\Delta v}{c} = \frac{\Delta \lambda}{\lambda} \quad [2]$$

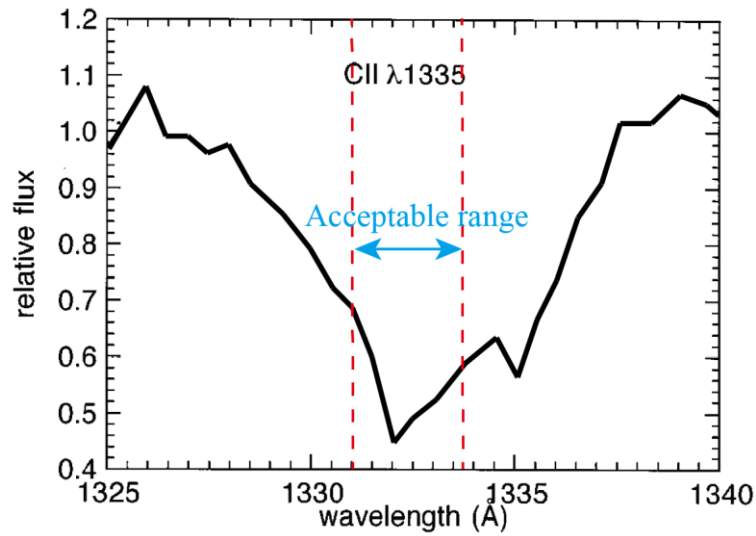


Figure 1 The C II $\lambda 1335$ absorption line with the acceptable range of the blueshifted wavelengths.

For the C II $\lambda 1335$ absorption line (Figure 1), we obtain

$$\begin{aligned} \Delta v_{\text{obs}}(\text{C II } \lambda 1335) &= 299792.458 \text{ km s}^{-1} \left(\frac{1332 - 1335}{1335} \right) \\ &= -673.7 \text{ km s}^{-1} \end{aligned} \quad [2]$$

Answers between -300 km s^{-1} to -900 km s^{-1} are acceptable with full marks.

The velocity offsets of the C II $\lambda 1335$ absorption line is -673.7 km s^{-1} . It means that the outflowing gas is going out of the galaxy at this speed, which is smaller than the escape velocity obtained in (b).

We conclude that the outflowing gas CANNOT escape from IRAS 0833+6517. [2]

(T10) GOTO

[25 marks]

The Gravitational-Wave Optical Transient Observer (GOTO) aims to carry out searches of optical counterparts of any Gravitational Wave (GW) sources within an hour of their detection by the LIGO and VIRGO experiments. The survey needs to cover a big area on the sky in a short time to search all possible regions constrained by the GW experiments before the optical burst signal, if any, fades away. The GOTO telescope array is composed of 4 identical reflective telescopes, each with 40-cm diameter aperture and f-ratio of 2.5, working together to image large regions of the sky. For simplicity, we assume that the telescopes' field-of-view (FoV) do not overlap with one another.

- a) Calculate the projected angular size per mm at the focal plane, i.e. plate scale, of each telescope. [5]

Solution

For any two points on the sky separated by small angular distance θ , at the focal plane of the telescope the distance is S

$$\tan \theta = \frac{S}{f}, \quad [0.5 \text{ Mark}]$$

where f is the focal length

for small angle θ

$$\tan \theta \approx \theta$$

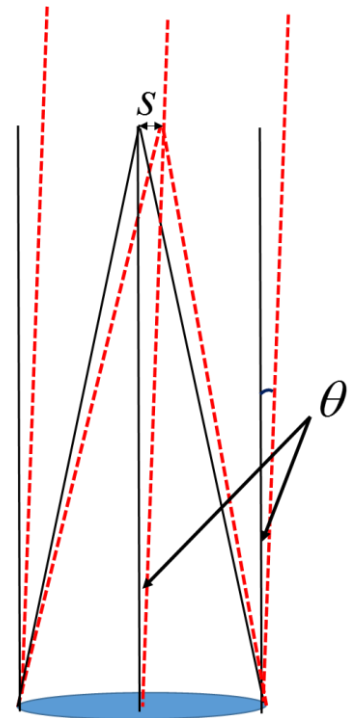
Then plate scale is

$$\frac{\theta}{S} = \frac{1}{f} \quad [0.5 \text{ Mark}]$$

$$f = \text{F-ratio} \times \text{Diameter} \quad [1 \text{ Mark}]$$

$$\text{Plate scale} = \frac{1}{\text{F-ratio} \times \text{Diameter}} \quad [1 \text{ Mark}]$$

If student correctly quote the formula for plate scale will also get the full mark for the above calculation



$$\text{Plate Scale} = \frac{1}{2.5 \times 40 \times 10^{-2} (\text{m}) \times 10^3 (\text{mm m}^{-1})} \times \frac{180 \times 3600 (\text{arcsec})}{\pi} \quad [1 \text{ Mark}]$$

$$= 206 \text{ arcsec/mm}$$

$$\text{Plate Scale} = 3.44 \text{ arcmin/mm} \quad [1 \text{ Mark}]$$

- b) If the zero-point magnitude (i.e. the magnitude at which the count rate detected by the detector is 1 count per second) of the telescope system is 18.5 mag, calculate the minimum time needed to reach 21 mag at Signal-to-Noise Ratio (SNR) = 5 for a point source. We first assume that the noise is dominated by both the Read-Out Noise (RON) at 10 counts/pixel and the CCD dark (thermal) noise (DN) rate of 1 count/pix/minute. The CCDs used with the GOTO have a 6-micron pixel size and gain (conversion factor between photo-electron and data count) of 1. The typical seeing at the observatory site is around 1.0 arcsec. [8]

The signal to Noise ratio is defined by

$$\text{SNR} = \frac{\text{Total Source Count}}{\sqrt{\sum_i \text{Noise}_i^2}} = \frac{\text{Total Source Count}}{\sqrt{\sigma_{\text{RON}}^2 + \sigma_{\text{DN}}^2 + \dots}},$$

$$\sigma_{\text{RON}} = \sqrt{N_{\text{pix}} \cdot \text{RON}^2}, \quad \sigma_{\text{DN}} = \sqrt{N_{\text{pix}} \cdot \text{DN} \cdot t},$$

where t is the exposure time.

Solution

From plate scale calculated in a) size of a star for typical seeing 1 arcsec

$$= \frac{1/60 \text{ arcmin}}{(3.44 \text{ arcmin/mm} \times 6 \times 10^{-3} \text{ mm})} \text{ pixels} \quad [1 \text{ Mark}]$$

$$= 0.8 \text{ pixel}$$

→ light from star is mostly contained within $N_{\text{pix}}=1$ pixel [1 Mark]

Alternative solution:

If a student gets the correct answer for 1 arcsec = 0.8 pixel but clearly explain that the chosen aperture size is $N \times$ “seeing” or a circular aperture with radius $R_{N_{\text{pix}}}$ (where $R_{N_{\text{pix}}}$ is not larger than $2 \times$ seeing) and calculate the **total number of pixels must be the closest round-up integer**.

And correctly use
$$\text{SNR} = \frac{\text{CR} \times t}{\sqrt{N_{\text{pix}} \text{RON}^2 + (N_{\text{pix}} \times \text{DN} \times t)}}$$

Then the student will be awarded full mark for the relevant parts.

This is also apply to part c)

From the definition of the Zero-point magnitude given in the question

$$m = -2.5 \log_{10}(\text{count rate}) + \text{Zero-Point} \quad [2 \text{ Mark}]$$

$$m = 21, \text{ ZP} = 18.5$$

$$\rightarrow \text{source count rate CR} = 10^{-(21-18.5)/2.5} = 0.1 \text{ per second} \quad [1 \text{ Mark}]$$

Signal-to-noise ratio

$$\text{SNR} = \frac{\text{CR} \times t}{\sqrt{\text{RON}^2 + (\text{DN} \times t)}}$$

$$(\text{CR} \times t)^2 = 25(\text{RON}^2 + (\text{DN} \times t)) \quad [1 \text{ Mark}]$$

Solve quadratic equation

$$t = \frac{25 \text{ DN} + \sqrt{(25 \text{ DN})^2 + 100 \text{ CR}^2 \text{ RON}^2}}{2 \text{ CR}^2} \quad [1 \text{ Mark}]$$

$$t = \frac{\frac{25}{60} + \sqrt{(\frac{25}{60})^2 + 100(0.1)^2 10^2}}{2 \times 0.1^2} = 521 \text{ s} = 8.68 \text{ minutes} \quad [1 \text{ Mark}]$$

- c) Normally when the exposure time is long and the source count is high then Poisson noise from the source is also significant. Determine the relation between SNR and exposure time in the case that the noise is dominated by Poisson noise of the source. Recalculate the minimum exposure time required to reach 21 mag with SNR=5 in part b) if Poisson noise is also taken into consideration. The Poisson noise (standard deviation) of the source is given by $\sigma_{\text{source}} = \sqrt{\text{Source Count}}$. In reality, there is also the sky background which can be important source of Poisson noise. For our purpose here, please ignore any sky background in the calculation. [6]

Solution

If we include the source Poisson noise,

$$\text{SNR} = \frac{\text{CR} \times t}{\sqrt{\text{RON}^2 + (\text{DN} \times t) + \text{CR} \times t}} \quad [1 \text{ Mark}]$$

For the Poisson noise dominated case

$$\text{SNR} \approx \frac{\text{CR} \times t}{\sqrt{\text{CR} \times t}} \quad [1 \text{ Mark}]$$

Hence $SNR \propto t^{\frac{1}{2}}$ [1 Mark]

Re-calculate exposure time

$$SNR = \frac{CR \times t}{\sqrt{RON^2 + (DN \times t) + CR \times t}},$$

$$(CR \times t)^2 = 25(RON^2 + (DN + CR)t) \quad [1 \text{ Mark}]$$

Solve quadratic equation

$$t = \frac{25(DN + CR) + \sqrt{25^2(DN + CR)^2 + 100CR^2RON^2}}{2CR^2} \quad [1 \text{ marks}]$$

$$t = 666 \text{ s} = 11.1 \text{ minutes} \quad [1 \text{ marks}]$$

- d) The typical localisation uncertainty of the GW detector is about 100 square-degrees and we would like to cover the entire possible location of any candidate within an hour after the GW is detected. Estimate the minimum side length of the square CCD needed for each telescope in terms of the number of pixels. You may assume that the time taken for the CCD read-out and the pointing change are negligible. [6]

Solution

In 1 hr, one can observe $\frac{60}{11.1} = 5.4$ [1 mark] $\rightarrow 5$ pointing
(round down to the nearest integer)

[1 mark]

Therefore, we need to cover $100 \text{ deg}^2 \rightarrow$ one pointing need to cover

$$\frac{100}{(5 \times 4 \text{ telescope})} = 5 \text{ deg}^2 \quad [1 \text{ mark}]$$

or $2.24 \times 2.24 \text{ deg}$ per pointing per telescope [1 mark]

$$\text{calculate size of each CCD} = \frac{2.24 \text{ deg} \times 60}{(3.4 \text{ arcmin/mm})(6 \times 10^{-3} \text{ mm})} \text{ pixels} \quad [1 \text{ mark}]$$

$= 6588.2 \text{ pixels} \rightarrow 6589 \text{ pixels}$ (pixel number need to be integer)

Therefore, we need minimum CCD size = $6589 \times 6589 \text{ pixels}$ [1 mark]

Answer 6588 pixel is also eligible for the mark

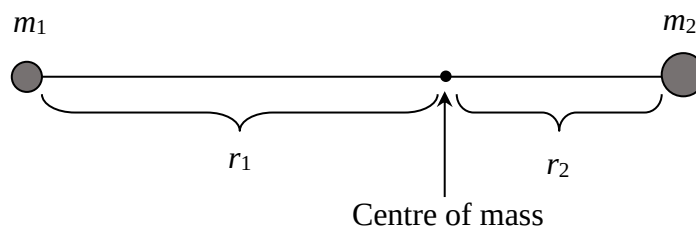
(T11) Mass of the Local Group

[50 marks]

The dynamics of M31 (Andromeda) and the Milky Way (MW) can be used to estimate the total mass of the Local Group (LG). The basic idea is that galaxies currently in a binary system were at approximately the same point in space shortly after the Big Bang. Besides, the mass of the local group is dominated by the masses of the MW and M31. Via Doppler shifts of the spectral lines, it was found that M31 is moving toward MW with a speed of 118 km s^{-1} . This may be surprising, given that most galaxies are moving away from each other with the general Hubble flow. The fact that the M31 is moving towards MW is presumably because their mutual gravitational attraction has eventually reversed their initial velocities. In principle, if the pair of galaxies is well-represented by isolated point masses, their total mass may be determined by measuring their separation, relative velocity and the time since the universe began. Kahn and Woltjer (1959) used this argument to estimate the mass in the LG.

In this problem we will follow this argument through our calculation as follows.

- a) Consider an isolated system with negligible angular momentum of two gravitating point masses m_1 and m_2 (as observed by an inertial observer at the centre of mass).



Write down the expression of the total mechanical energy (E) of this system in mathematical form connecting m_1 , m_2 , r_1 , r_2 , v_1 , v_2 , and the universal gravitational constant G , where v_1 and v_2 are the radial velocities of m_1 and m_2 , respectively. **[5]**

Solution:

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{G m_1 m_2}{r_1 + r_2} = E. \text{ Note that the total energy is negative quantity.} \quad \mathbf{[5]}$$

Detailed Marking Scheme:

Students get 1.5 marks for each kinetic energy term.

Students get 1.0 mark for correct sign in front of the potential energy term and another 1.0 mark for the correct expression of the potential energy.

- b) Re-write the equation in a) in terms of r , v , μ , M , and G , where $r \equiv r_1 + r_2$ is the separation distance between m_1 and m_2 , v is the changing rate of the separation distance,

$\mu \equiv \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass of the system, and $M \equiv m_1 + m_2$ is the total mass of the system. [10]

Solution:

From $m_1 r_1 = m_2 r_2$ and $r_1 + r_2 = r$, we have $r_1 = \frac{m_2}{m_1 + m_2} r$ and $r_2 = \frac{m_1}{m_1 + m_2} r$.

Substituting them into the solution obtained in a), we get

$$\begin{aligned} \frac{1}{2} v^2 - \frac{G(m_1 + m_2)}{r} &= \frac{E}{m} \\ v^2 - \frac{2GM}{r} &= \frac{2E}{m} \end{aligned}$$

Therefore, $E = \frac{\mu}{2} \left(v^2 - \frac{2GM}{r} \right)$ [10]

Detailed Marking Scheme:

Students get 2.0 marks for using the conservation of momentum or center of mass to get $m_1 r_1 = m_2 r_2$.

Students get 1.0 mark for each of the relationship $r_1 = \frac{m_2}{m_1 + m_2} r$ or $r_2 = \frac{m_1}{m_1 + m_2} r$. (total 2.0 marks)

Students get 1.0 mark each for the relationships between v and v_1 , and between v and v_2 . (total 2.0 marks)

For correct substitution of suitable variables into the expression for E , students get 2.0 marks.

Final expression for E is worth 2.0 marks.

c) Show that the equation in b) yields

$$v^2 = (2GM) \left(\frac{1}{r} - \frac{1}{r_0} \right), \text{ where } r_0 \text{ is a new constant.}$$

Find r_0 in terms of μ , M , G and E . [5]

Solution:

From b), we have $E = \frac{\mu}{2} \left(v^2 - \frac{2GM}{r} \right)$ or $v^2 - \frac{2GM}{r} = \frac{2E}{m}$

$$v^2 = \frac{2GM}{r} + \frac{2E}{m}$$

$$= (2GM) \left\{ \frac{1}{r} + \frac{1}{\left(\frac{GM\mu}{E} \right)} \right\}$$

$$= (2GM) \left\{ \frac{1}{r} - \frac{1}{r_0} \right\}, \text{ where } r_0 \equiv -\frac{GM\mu}{E}$$

Thus

$$r_0 = -\frac{GMm}{E} \quad [5]$$

Detailed Marking Scheme:

Students get 4.0 marks for correct algebra and obtain the expression in the correct form.

By correctly identifying that $r_0 = -\frac{GMm}{E}$, students get 1.0 mark.

The solution of the equation in b) is given below in parametric form, under the initial condition $r = 0$ at $t = 0$:

$$r(\theta) = \frac{r_0}{2} (1 - \cos \theta),$$

$$t(\theta) = \left(\frac{r_0^3}{8GM} \right)^{\frac{1}{2}} (\theta - \sin \theta),$$

where θ is in radians.

d) From the above parametric equations, show that an expression for $\frac{vt}{r}$ is

$$\frac{vt}{r} = \frac{(\sin q)(q - \sin q)}{(1 - \cos q)^2}. \quad [10]$$

Solution:

Substituting $r(\theta)$ in the equation of part c):

$$v^2 = 2GM \left(\frac{1}{r} - \frac{1}{r_0} \right) = 2GM \left(\frac{1}{\frac{1}{2}r_0(1-\cos\theta)} - \frac{1}{r_0} \right) = \frac{2GM}{r_0} \left(\frac{1+\cos\theta}{1-\cos\theta} \right)$$

$$v^2 = \frac{2GM}{r_0} \frac{(1-\cos^2\theta)}{(1-\cos\theta)^2} = \frac{2GM}{r_0} \frac{\sin^2\theta}{(1-\cos\theta)^2} \Rightarrow v(\theta) = \left(\frac{2GM}{r_0} \right)^{1/2} \frac{\sin\theta}{1-\cos\theta}$$

Combining $v(\theta)$, $t(\theta)$, and $r(\theta)$:

$$\frac{v(\theta)t(\theta)}{r(\theta)} = \frac{\left(\frac{2GM}{r_0} \right)^{1/2} \frac{\sin\theta}{1-\cos\theta} \cdot \left(\frac{r_0^3}{8GM} \right)^{1/2} (\theta - \sin\theta)}{\frac{r_0}{2}(1-\cos\theta)} = \frac{2 \left(\frac{2GM r_0^3}{r_0 8GM} \right)^{1/2} (\sin\theta)(\theta - \sin\theta)}{(1-\cos\theta)^2}$$

$$\frac{vt}{r} = \frac{(\sin\theta)(\theta - \sin\theta)}{(1-\cos\theta)^2}$$

Alternative solution:

From $r(\theta) = \frac{r_0}{2}(1 - \cos\theta)$, we have

$$v = \left(\frac{r_0}{2} \sin\theta \right) \omega$$

From $t(\theta) = \left(\frac{r_0^3}{8GM} \right)^{1/2} (\theta - \sin\theta)$, we have

$$1 = \left(\frac{r_0^3}{8GM} \right)^{1/2} (1 - \cos\theta) \omega$$

$$\therefore \frac{vt}{r} = \frac{\left(\frac{r_0}{2} \sin\theta \right) \left(\frac{r_0^3}{8GM} \right)^{1/2} (\theta - \sin\theta)}{\left(\frac{r_0^3}{8GM} \right)^{1/2} (1 - \cos\theta) \frac{r_0}{2} (1 - \cos\theta)} = \frac{(\sin\theta)(\theta - \sin\theta)}{(1 - \cos\theta)^2}. \quad [10]$$

Detailed Marking Scheme:

Student gets 3.0 marks for substituting $r(\theta)$ in v^2 and obtaining $\frac{2GM}{r_0} \left(\frac{1+\cos\theta}{1-\cos\theta} \right)$.

Student gets 3.0 marks for using trigonometry to simplify the expression for $v(\theta)$.

For the correct derivation of the final answer, student gets 4.0 marks.

- e) Now we consider m_1 and m_2 as the MW and M31 respectively. The current values of v and r are $v = -118 \text{ km s}^{-1}$ and $r = 710 \text{ kpc}$, and t may be taken to be the age of the Universe (13700 million years). Find θ using numerical iteration. [10]

Solution:

$$v = -118 \text{ km s}^{-1} = -118 \times 10^3 \text{ m s}^{-1}$$

$$r = 710 \text{ kpc} = 2.19 \times 10^{22} \text{ m}$$

$$t = 13700 \times 10^6 \text{ years} = 4.3233 \times 10^{17} \text{ s}$$

$$\therefore \frac{vt}{r} = -\frac{118 \times 10^3 \times 4.3233 \times 10^{17}}{2.1908 \times 10^{22}} = -2.33 = \frac{(\sin \theta)(\theta - \sin \theta)}{(1 - \cos \theta)^2} \quad [5]$$

The negative value of the left-hand side of this equation implies that θ is greater than π .

θ	$(\sin \theta)(\theta - \sin \theta) / (1 - \cos \theta)^2$
$200^\circ = 3.49 \text{ radians}$	-0.348
$210^\circ = 3.67 \text{ radians}$	-0.598
$240^\circ = 4.19 \text{ radians}$	-1.95
$244^\circ = 4.26 \text{ radians}$	-2.24
$245^\circ = 4.28 \text{ radians}$	-2.32 ←
$246^\circ = 4.29 \text{ radians}$	-2.40
$250^\circ = 4.36 \text{ radians}$	-2.77

This gives the value of
 $245^\circ = 4.28$
radians

[5]

Detailed Marking Scheme:

Students get 5.0 marks if numerical value and sign (2.5 marks for each) are both correct.

Students get 5.0 marks if the obtained angle (θ) is within ± 1 degree (or equivalent in radian), or 2.0 marks for answer within ± 3 degrees (or equivalent in radian).

- f) Use the value of θ in e) to obtain the value of $r_{\max}$. Hence also obtain the value of M in solar masses. [10]

Solution:

From $r(\theta) = \frac{r_0}{2}(1 - \cos \theta)$, we have $r_{\max} = r_0$, and hence

$$r_{\max} = \frac{2r(\theta)}{1 - \cos \theta} = \frac{2 \times 710 \text{ kpc}}{1 - \cos 245^\circ} = 998 \text{ kpc} \quad [5]$$

(This is the maximum separation between the Milky-Way Galaxy and Andromeda Galaxy; they started to move towards each other afterwards.)

The value of M can be calculated from the relation $t(\theta) = \left(\frac{r_0^3}{8GM} \right)^{\frac{1}{2}} (\theta - \sin \theta)$.

$$M = \left(\frac{r_{\max}^3}{8G} \right) \left\{ \frac{\theta - \sin \theta}{t(\theta)} \right\}^2.$$

From the maximum separation of $r_{\max} = 998 \text{ kpc} = 3.0795 \times 10^{22} \text{ m}$,
 $\theta = 245^\circ = 4.276 \text{ radians}$, $t = t(\theta) = 13700 \times 10^6 \text{ years} = 4.3233 \times 10^{17} \text{ s}$, and
 $G = 6.67 \times 10^{-11} \text{ N m}^2 \text{ kg}^{-2}$, we obtain

$$\therefore M = 7.86 \times 10^{42} \text{ kg} = \frac{7.86 \times 10^{42}}{1.99 \times 10^{30}} M_\odot = 3.95 \times 10^{12} M_\odot \quad [5]$$

The estimated number of stars, most of which are of less than a solar mass, for Andromeda and Our Galaxy are 4×10^{11} and 2×10^{11} respectively. This implies that most of mass of the system is DARK.

Detailed Marking Scheme:

First part which is counting for 5.0 marks for determining $r_{\max}$ is detailed as follow:
 Students get 2.0 marks for realizing that $r_{\max} = r_0$.
 Students get 1.0 mark by re-arranging $r_{\max}$ in term of $r(\theta)$ and θ .
 Students get 1.0 mark if they substitute the correct values given in the question.
 Students get 1.0 mark for correct numerical answer.

Second part that is worth 5.0 marks for determining M is detailed as follow:
 Students get 2.0 marks for successfully expressing M in term of known variables.

Students get 1.0 mark by substituting the correct numerical values given in the question into the expression from previous step.
Students get 2.0 marks for obtaining the correct answer in requested unit.

Note that

$$v = \frac{dr}{dt} = + (2GM)^{1/2} \left(\frac{r_0 - r}{r_0 r} \right)^{1/2} \text{ for } r_0 > r$$

$$\int (2GM)^{1/2} dt = + \int \left(\frac{r_0 r}{r_0 - r} \right)^{1/2} dr$$

Putting $\frac{r}{r_0} \equiv \sin^2 \phi$

$$t = \left(\frac{r_0^3}{8GM} \right)^{1/2} \{ 2\phi - \sin 2\phi \} + D$$

We choose the initial condition such that m_1 and m_2 are close together, that is that $r = 0$ at $t = 0$. This implies that $\phi = 0$ at $t = 0$, hence D must be zero.

$$t = \left(\frac{r_0^3}{8GM} \right)^{1/2} (\theta - \sin \theta), \quad \theta \equiv 2\phi.$$

And from $r = r_0 \sin^2 \phi = \frac{1}{2} r_0 (1 - \cos 2\phi)$, we have:

$$r = \frac{1}{2} r_0 (1 - \cos \theta).$$

Equation (5) and (6) form the parametric solution of equation (4).

$$r_{\max} = r(\theta = \pi) = r_0$$

(T12) Shipwreck

[40 mark]

You are shipwrecked on an island. Fortunately, you are still wearing a watch that is set to Bangkok time. You also have a compass, an atlas and a calculator. You are initially unconscious, but wake up to find it has recently become dark. Unfortunately it is cloudy. An hour or so later you see Orion through a gap in the clouds. You estimate that the star “Rigel” is about 52.5° above the horizon and with your compass you find that it has an astronomical azimuth of 109° . Your watch says 01:00 on the 21st November 2017. You happen to remember from your astronomy class that Greenwich Sidereal Time (GST) at 00h UT 1st January 2017 is about 6h 43min and that R.A. and Dec. of Rigel are 5h 15min and $-8^\circ 11'$, respectively.

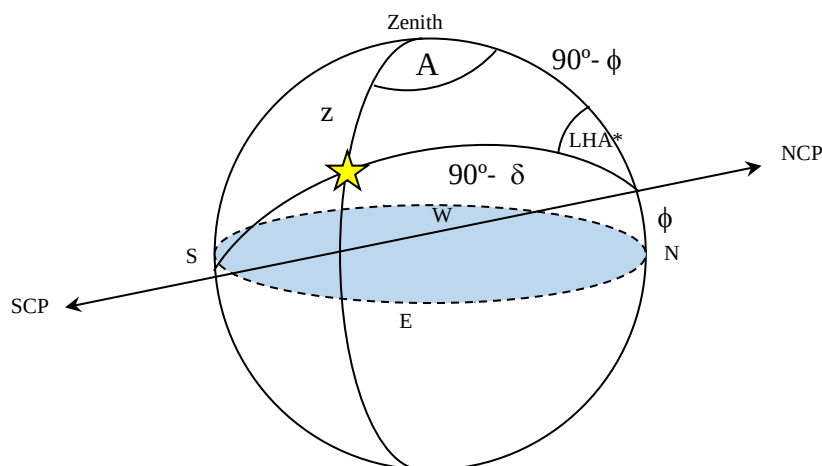
a) Find the Local Hour Angle (LHA) of Rigel.

[10]

Solution:

Draw a correct celestial sphere

[4.0]



$$\frac{\sin LHA^*}{\sin z} = \frac{\sin A}{\sin(90^\circ - \delta)} \quad [2.0]$$

$$LHA^* = \sin^{-1} \left(\frac{\sin A \sin z}{\sin(90^\circ - \delta)} \right)$$

$$= \sin^{-1} \left(\frac{\sin 109^\circ \sin 37.5^\circ}{\sin 98.18^\circ} \right) \quad [2.0]$$

$$LHA^* = 35.56^\circ = 2h 22 \text{ min}$$

$$LHA = 24h - 2h 22 \text{ min} = 21h 38 \text{ min} \quad [2.0]$$

b) Find the current Greenwich sidereal time (GST).

[10]

Solution:

Bangkok time of 01:00 21st November is Universal Time (UT) 18:00 20th November

Day number + Time = 323 day 18 h [1.0]

GST = GST of 1st Jan + The angle that Y moves away from 1st Jan [2.0]

$$\text{GST} = 6\text{h } 43\text{min} + \left(323 \frac{18}{24} \text{ days} \right) \left(\frac{1}{365.2422} \right) \times 24 \frac{\text{h}}{\text{day}} + \left[18\text{h} \times \left(\frac{24}{23.9344} \right) \right] \quad [5.0]$$

$$\text{GST} = 6\text{h } 43\text{min} + 21\text{h } 16 \text{ min} + 18\text{h } 3\text{min}$$

$$\text{GST} = 46\text{h } 2\text{min}$$

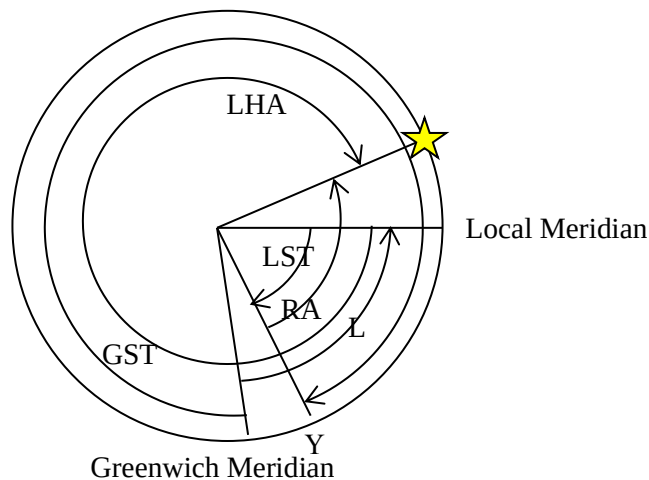
$$\text{GST} = 21\text{h } 58\text{min} \quad [2.0]$$

Any other correct alternative methods that provide the correct answer also acceptable.

c) Find the longitude of the island.

[5]

Solution:



$$24\text{h} - \text{LHA} = \text{RA} - \text{LST}$$

$$\text{LST} = \text{RA} - 24\text{h} + \text{LHA}$$

$$\text{L} = \text{RA} + \text{LHA} - \text{GST}$$

$$\text{L} = 5\text{h } 15\text{min} + 21\text{h } 38 \text{ min} - 21\text{h } 58\text{min} = 4\text{h } 55\text{min} \quad [3.0]$$

$$L = 73.75^\circ \text{ E}$$

[2.0]

d) Find, accurately to the nearest arcminute, the Latitude of the island.

[15]

Solution:

$$\cos(90^\circ - \delta) = \cos z \cos(90^\circ - \phi) + \sin z \sin(90^\circ - \phi) \cos A$$

$$\sin \delta = \sin(h) \sin \phi + \cos(h) \cos A \cos \phi \quad [5.0]$$

$$\alpha = \sin \delta, \beta = \sin(h), \gamma = \cos(h) \cos A, x = \sin \phi$$

$$\alpha = \beta x + \gamma \sqrt{1 - x^2}$$

$$(\beta^2 + \gamma^2)x^2 - 2\alpha\beta x + (\alpha^2 - \gamma^2) = 0 \quad [1.0]$$

$$x = \frac{\alpha\beta \pm \gamma \sqrt{\gamma^2 + \beta^2 - \alpha^2}}{\beta^2 + \gamma^2} \quad [1.0]$$

$$\sin \phi = \frac{\sin \delta \sin(h) \pm \cos(h) \cos A \sqrt{(\cos(h) \cos A)^2 + \sin^2(h) - \sin^2 \delta}}{\sin^2(h) + (\cos(h) \cos A)^2} \quad [1.0]$$

$$\phi = -24.05^\circ, 4.01^\circ \quad [2.0]$$

$$\text{For } \phi = -24.05^\circ \rightarrow A = 71.0^\circ \quad [2.0]$$

$$\text{For } \phi = 4.01^\circ \rightarrow A = 109^\circ \quad [2.0]$$

$$\text{The Latitude of the island is } 4^\circ 1' \text{ N} \quad [1.0]$$

Alternative Solutions

b)

Solution:

$$- \text{LMT (Local Mean Time)} \rightarrow \text{UT} \quad [1.0]$$

Calculate sidereal day using given tropical year

$$\gamma = \frac{\text{Sidereal time}}{\text{Solar time}} = \frac{24^h \times \left(1 + \frac{1}{T_{\text{tropic}}}\right)}{24^h}$$

$$= 1 + \frac{1}{T_{\text{tropic}}} = 1.002737909 \quad [3.0]$$

On the 21st November 2017, GST can be calculated from GST on 1st January 2017

$$\text{Day number} + \text{Time} = 323 \text{ d } 18 \text{ h} \quad [2.0]$$

$$\text{GST} = \text{GST}_0 + \Delta\theta = \text{GST}_0 + \gamma \Delta T \quad [2.0]$$

$$\Delta\theta = \left(323^{\text{d}} \frac{18}{24}\right) \times 1.002737909$$

$$= 324^{\text{d}}.6363981 = 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$\text{GST} = \text{GST}_0 + 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$= 6^{\text{h}}43^{\text{m}} + 15^{\text{h}}16^{\text{m}}24^{\text{s}}.8$$

$$= 21^{\text{h}}59^{\text{m}}24^{\text{s}}.8 \quad [2.0]$$

d)

Solution:

From Cosine law:

$$\sin \delta = (\cos z) \sin \phi + (\sin z \cos A) \cos \phi \quad [5.0]$$

$$\cos z = (\sin \delta) \sin \phi + (\cos \delta \cos \text{LHA}^*) \cos \phi$$

Correct calculation steps that can lead to the final answers. [5.0]

Final answers

$$\sin \phi = 0.0698$$

$$\phi = 4.00^\circ$$

OR

$$\cos \phi = 0.9976$$

$$\phi = 4.00^\circ$$

[5.0]

(T13) Exomoon

[60 marks]

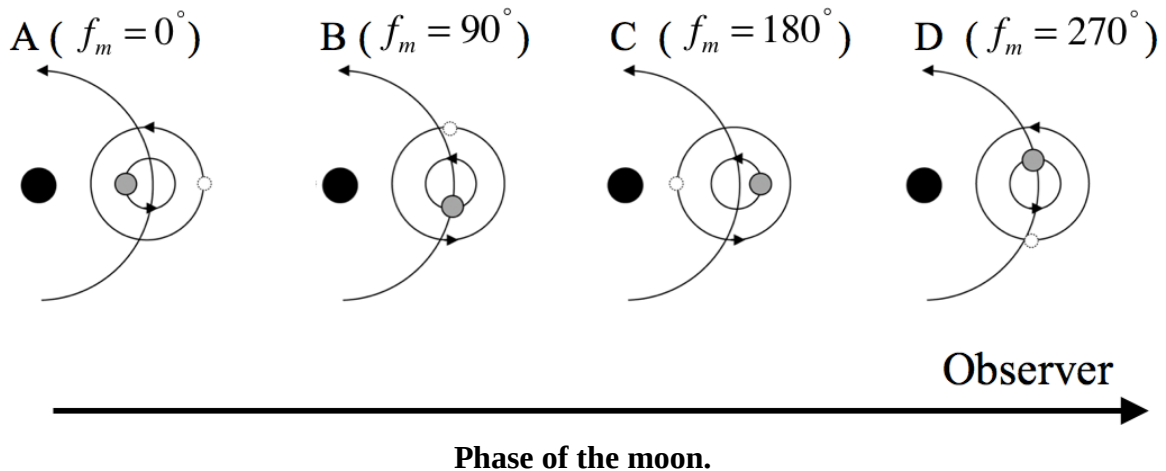
Exomoons are natural satellites of exoplanets. The gravitational influence of such a moon will affect the position of the planet relative to the planet-moon barycentre, resulting in Transit Timing Variations (σ_{TTV} , TTVs) as the observed transit of the planet occurs earlier or later than the predicted time of transit for a planet without a moon.

The motion of the planet around the planet-moon barycentre will also induce Transit Duration Variations (σ_{TDV} , TDVs) as the observed transit duration is shorter or longer than the predicted transit duration for a planet without a moon.

We will consider edge-on circular orbits with the following parameters

- M_p is the planet mass
- M_m is the moon mass
- P_p is the planet-moon barycentre's period around the host star
- P_m is the moon's period around the planet
- a_p is the distance of the planet-moon barycentre to the star
- a_m is the distance of the moon to the planet-moon barycentre
- f_m is the moon phase, $f_m = 0^\circ$ when the moon is in opposition to the star
- τ is the mean transit duration of the planet (as if it has no moon)

We will consider only orbit of a prograde moon with an orbit in the same plane as the planet's orbit. Example phases of the moon, as observed by distant observers, are shown in the figure below.



Black, grey and white circles represent the star, planet and moon, respectively.

- a. We define $\sigma_{TTV} \equiv t_m - t$ where t is the predicted transit time without the moon, and t_m is the observed transit time with the moon. Show that

$$\sigma_{TTV} = \left[\frac{a_m M_m P_p}{2\pi a_p M_p} \right] \sin(f_m)$$

A positive value of σ_{TTV} indicates that the transit occurs later than the predicted time of transit for a planet without a moon. [10]

Solution:

From centre of mass for planet-moon system, the distance of the moon to the planet-moon barycenter can be written as,

$$a_m = \frac{M_p}{M_m} a_{pb} \quad [2.0]$$

where a_{pb} is distance of the planet to the planet-moon barycentre.

For the observers on the Earth, projection distance of the planet to the planet-moon barycenter is

$$a_{proj} = a_{pb} \sin(f_m) \quad [3.0]$$

From angular velocity, TTV signal can be calculated from

$$\sigma_{TTV} \equiv t_m - t$$

$$\sigma_{TTV} = \frac{\theta_{trans}}{\omega_p} \quad [1.0]$$

$$\sigma_{TTV} = \left(\frac{a_{proj}}{a_p} \right) \bigg/ \left(\frac{2\pi}{P_p} \right) \quad [2.0]$$

$$\sigma_{TTV} = \frac{a_{proj} P_p}{a_p 2\pi}$$

$$\sigma_{TTV} = \left[\frac{a_m M_m P_p}{2\pi a_p M_p} \right] \sin(f_m) \quad [2.0]$$

- b. $\sigma_{TDV} \equiv \tau_m - \tau$ where τ is the predicted transit duration without the moon, and τ_m is the observed transit duration with the moon. We can assume that the planet's velocity around the star is much bigger than the moon's velocity around the planet-moon barycentre, and also the moon does not change phase during the transit. Show that

$$\sigma_{TDV} = \tau \left[\frac{P_p M_m a_m}{P_m M_p a_p} \right] \cos(f_m)$$

A positive value of σ_{TDV} indicates that the transit duration is longer than the predicted transit duration without a moon. [13]

Solution:

The velocity of planet-moon barycenter around the star is

$$v_p = \frac{2\pi}{P_p} a_p \quad [1.0]$$

The velocity of planet around the planet-moon barycentre is

$$v_{pb} = \frac{2\pi}{P_m} a_{pb} \quad [1.0]$$

Therefore, the transverse velocity of planet around the planet-moon barycentre is

$$v_{trans} = -v_{pb} \cos(f_m) \quad [3.0]$$

$$v_{trans} = -\frac{2\pi M_m}{P_m M_p} a_m \cos(f_m)$$

Average transit duration can be written as

$$\tau = \frac{D}{v_p} \quad [2.0]$$

where D is the distance of the planet has to cross in order to complete the transit. The transit duration with exomoon can be written as

$$\tau_m = \frac{D}{v_p + v_{trans}} \quad [2.0]$$

Therefore, the TDV signal is

$$\sigma_{TDV} \equiv \tau_m - \tau = \frac{\tau v_p}{v_p + v_{trans}} - \tau \quad [1.0]$$

$$\sigma_{TDV} = \tau \left[\frac{-v_{trans}}{v_p + v_{trans}} \right]$$

In case $v_p \gg v_{trans}$

$$\sigma_{TDV} = -\tau \left[\frac{v_{trans}}{v_p} \right] \quad [2.0]$$

$$\sigma_{TDV} = \tau \left[\frac{P_p M_m a_m}{P_m M_p a_p} \right] \cos(f_m) \quad [1.0]$$

An exoplanet is observed transiting a main-sequence solar type star ($1 M_\odot$, $1 R_\odot$, Spectral class: G2V). The planet has an edge-on circular orbit with a period of 3.50 days. From the observational data, the planet has a mass of $120 M_\oplus$ and a radius of $12 R_\oplus$. The observed relation between σ_{TTV}^2 and σ_{TDV}^2 can be written as

$$\sigma_{TDV}^2 = -0.7432 \sigma_{TTV}^2 + 1.933 \times 10^{-8} \text{ days}^2$$

- c. Assume that the moon's mass is much smaller than the planet's mass. Find the mean transit duration of the planet (τ) in days. [6]

Solution:

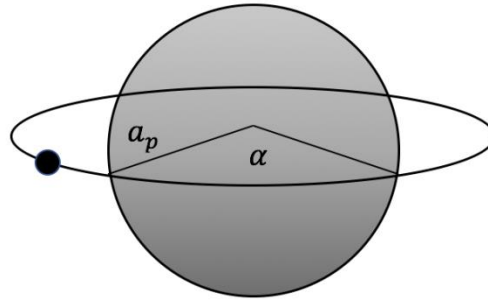
From Kepler's third's law, assume that the moon mass is much smaller than the planet mass, the distance of the planet-moon barycentre to the star is

$$P^2 = \frac{4\pi^2}{G(M_* + M_p)} a_p^3 \quad [1.0]$$

$$(3.5 \times 86400)^2 = \frac{4\pi^2}{6.67 \times 10^{-11} \times (1.99 \times 10^{30} + 120 \times 5.98 \times 10^{24})} a_p^3$$

$$a_p = 6.75 \times 10^9 \text{ m}$$

The mean transit duration of the planet can be calculated from the transit duration of exoplanet without exomoon.



$$\tau = P \frac{\alpha}{2\pi} \quad [1.0]$$

$$\tau = \frac{P}{2\pi} \times 2 \sin^{-1} \left(\frac{R_* + R_p}{a_p} \right) \quad [2.0]$$

$$\tau = \frac{3.5}{\pi} \sin^{-1} \left(\frac{6.96 \times 10^8 + 12 \times 6.38 \times 10^6}{6.75 \times 10^9} \right) \text{ days} \quad [2.0]$$

$$\tau = 0.128 \text{ days}$$

- d. Find the moon's period (P_m) in days [7]

Solution:

The TTV signal is 90 degrees out of phase with the TDV signal in edge-on circular orbit system. Therefore, the relation between the TTV and TDV signals is,

$$\sigma_{TDV}^2 = - \left(\frac{2\pi\tau}{P_m} \right)^2 \sigma_{TTV}^2 + \tau^2 \left(\frac{a_m M_m P_p}{a_p M_p P_m} \right)^2 \quad [3.0]$$

The observed relation between σ_{TTV}^2 and σ_{TDV}^2 can be written as

$$\sigma_{TDV}^2 = -0.7432\sigma_{TTV}^2 + 1.933 \times 10^{-8} \text{ days}^2$$

Therefore

$$-\left(\frac{2\pi\tau}{P_m}\right)^2 = -0.7432 \quad [2.0]$$

$$P_m = 0.933 \text{ days} \quad [2.0]$$

- e. Estimate the distance of the moon to the planet-moon barycentre (a_m) in units of Earth radii. Also find the moon mass (M_m) in units of Earth mass. [7]

Solution:

From Kepler third law, assuming that moon is small and can be neglected, the distance of the moon to the planet-moon barycentre is

$$P_m^2 = \frac{4\pi^2}{GM_p} a_m^3 \quad [1.0]$$

$$(0.933 \times 86400)^2 = \frac{4\pi^2}{6.67 \times 10^{-11} \times (120 \times 5.98 \times 10^{24})} a_m^3$$

$$a_m = 1.99 \times 10^8 \text{ m} = 31.2 R_{\oplus} \quad [2.0]$$

From the observed relation between σ_{TTV}^2 and σ_{TDV}^2 , the moon mass is

$$\tau^2 \left(\frac{a_m M_m P_p}{a_p M_p P_m} \right)^2 = 1.933 \times 10^{-8} \text{ days}^2 \quad [2.0]$$

$$(0.128)^2 \left(\frac{(1.99 \times 10^8) \times M_m \times (3.5)}{(6.75 \times 10^9) \times (120) \times (0.933)} \right)^2 = 1.933 \times 10^{-8} \text{ days}^2$$

$$M_m = 1.18 M_{\oplus} \quad [2.0]$$

- f. The Hill sphere is a region around a planet within which the planet's gravity dominates. The radius of the Hill sphere can be written as

$$R_h = a_p \sqrt[3]{\frac{M_p}{3M_*}}$$

where M_* is the host star mass.

Find the value of the constant x (Hint: for a massive host star, the radius of the Hill sphere of the system is approximately equal to the distance between the planet and the Lagrange point L_1 or L_2). Then find the radius of the Hill sphere of this planetary system in units of Earth radii. [11]

Solution:

For planet-moon system, the angular velocity of barycenter of planet-moon system is

$$F_{\text{centrifugal-bary}} = F_{\text{star-bary}}$$

$$(M_p + M_m)\omega_p^2 a_p = \frac{GM_*(M_p + M_m)}{a_p^2}$$

$$\omega_p^2 = \frac{GM_*}{a_p^3} \quad [2.0]$$

For exomoon at Lagrange point L_1 or L_2 ,

$$F_{\text{centrifugal-moon}} = F_{\text{star-moon}} \pm F_{\text{planet-moon}} \quad [1.0]$$

$$M_m \omega_p^2 (a_p \pm a_m) = \frac{GM_* M_m}{(a_p \pm a_m)^2} \pm \frac{GM_p M_m}{(a_{pb} + a_m)^2} \quad [3.0]$$

$$\frac{M_*(a_p \pm a_m)}{a_p^3} = \frac{M_*}{(a_p \pm a_m)^2} \pm \frac{M_p}{(a_{pb} + a_m)^2}$$

Since $M_p \gg M_m$ and $a_{pb} \ll a_m$

$$\frac{M_*(a_p \pm a_m)}{a_p^3} = \frac{M_*}{(a_p \pm a_m)^2} \pm \frac{M_p}{a_m^2} \quad [1.0]$$

$$M_*(a_p \pm a_m)^3 a_m^2 = M_* a_p^3 a_m^2 \pm M_p a_p^3 (a_p \pm a_m)^2$$

$$M_*(\pm 3a_p^2 a_m^3 + 3a_p a_m^4 \pm a_m^5) = \pm M_p a_p^3 (a_p \pm a_m)^2$$

$$\text{For } a_p \gg a_m, M_*(3a_p^2 a_m^3) \gg M_p a_p^5 \quad [1.0]$$

$$a_m^3 = \frac{M_p}{3M_*} a_p^3$$

$$a_m = a_p \sqrt[3]{\frac{M_p}{3M_*}}$$

$$x = 3 \quad [1.0]$$

The radius of the Hill sphere of this planetary system is

$$R_h = a_p \sqrt[3]{\frac{M_p}{3M_*}}$$

$$R_h = 6.75 \times 10^9 \times \sqrt[3]{\frac{120 \times 5.98 \times 10^{24}}{3 \times 1.99 \times 10^{30}}} \text{ m}$$

$$R_h = 3.33 \times 10^8 \text{ m} = 52.2 R_{\oplus} \quad [2.0]$$

- g. The Roche limit is the minimum orbital radius at which a satellite can orbit without being torn apart by tidal forces and take the Roche limit as

$$R_r = 1.26 R_p \sqrt[3]{\frac{\rho_p}{\rho_m}}$$

where ρ_p and ρ_m are the density of the planet and moon, respectively. R_p is planet radius. Assuming that the moon is a rocky moon with Earth's density, find the Roche limit of the system. [3]

Solution:

From Roche limit equation

$$R_r = 1.26 R_p \sqrt[3]{\frac{\rho_p}{\rho_m}}$$

$$R_r = 1.26 \times 12 \times \sqrt[3]{\frac{120 / (12)^3}{1}} R_{\oplus}$$

$$R_r = 6.21 R_{\oplus}$$

[3.0]

- h. Does the moon have a stable orbit? [3]

Solution:

Stable orbit because the distance between the moon and planet-moon barycentre is between Roche limit and Hill sphere radius ($R_r < a_m < R_H$) [3.0]

Mark only if students get the answers for e) & g) & h).
Students obtain full mark if answer "YES" for $R_r < a_m < R_H$, and vice versa,

(T1) **True or False**

Determine if each of the following statements is True or False. In the Summary Answersheet, tick the correct answer (TRUE / FALSE) for each statement. No justifications are necessary for this question.

- (T1.1) In a photograph of the clear sky on a Full Moon night with a sufficiently long exposure, the colour of the sky would appear blue as in daytime. 2

Solution:

☐ T

The colour of the clear sky during night is the same as during daytime, since the spectrum of sunlight reflected by the Moon is almost the same as the spectrum of sunlight. Only the intensity is lower.

2.0

- (T1.2) An astronomer at Bhubaneswar marks the position of the Sun on the sky at 05:00 UT every day of the year. If the Earth's axis were perpendicular to its orbital plane, these positions would trace an arc of a great circle. 2

Solution:

☐ T

If the Earth's axis were perpendicular to its orbital plane, the celestial equator will coincide with ecliptic and the Sun will remain along the celestial equator every day. However, as the Earth's orbit is elliptical, the true sun would still lead or lag mean sun by a few minutes on different days of year.

2.0

- (T1.3) If the orbital period of a certain minor body around the Sun in the ecliptic plane is less than the orbital period of Uranus, then its orbit must necessarily be fully inside the orbit of Uranus. 2

Solution:

☐ F

The semi-major axis of the orbit of the body will be less than that of Uranus. However the minor body's orbit may have a high eccentricity, in which case it may go outside that of Uranus.

2.0

- (T1.4) The centre of mass of the solar system is inside the Sun at all times. 2

Solution:

☐ F

The centre of mass of Sun-Jupiter pair is just outside the Sun. Thus, if all gas giants are on same side of the Sun, the centre of mass of Solar system is definitely outside the Sun.

2.0

- (T1.5) A photon is moving in free space. As the Universe expands, its momentum decreases. 2

Solution:

☐ T

For photons the wavelength increases when the Universe expands.

2.0

(T2) **Gases on Titan**

10

Gas particles in a planetary atmosphere have a wide distribution of speeds. If the r.m.s. (root mean square) thermal speed of particles of a particular gas exceeds $1/6$ of the escape speed, then most of that gas will escape from the planet. What is the minimum atomic weight (relative atomic mass), $A_{\min}$, of an ideal monatomic gas so that it remains in the atmosphere of Titan?

Given, mass of Titan $M_T = 1.23 \times 10^{23}$ kg, radius of Titan $R_T = 2575$ km, surface temperature of Titan $T_T = 93.7$ K.

Solution:

As the gas is monatomic,

$$\frac{3}{2}k_B T_T \approx \frac{1}{2}m_g v_{\text{rms}}^2$$

$$3k_B T_T \approx \frac{M_g}{N_A} v_{\text{rms}}^2$$

$$\therefore v_{\text{rms}} \approx \sqrt{\frac{3k_B N_A T_T}{M_g}}$$

4.0

50% deduction if $3/2$ pre-factor is not used and $1/2$ or 1 are used instead.

Full credit if students writes the relation for v_{rms} directly.

To remain in atmosphere,

$$v_{\text{rms}} < \frac{v_{\text{esc}}}{6} = \frac{1}{6} \sqrt{\frac{2GM_T}{R_T}}$$

$$\sqrt{\frac{3k_B N_A T_T}{M_g}} < \sqrt{\frac{GM_T}{18R_T}}$$

$$\therefore M_g > \frac{54k_B N_A T_T R_T}{GM_T}$$

$$> \frac{54 \times 1.381 \times 10^{-23} \times 6.022 \times 10^{23} \times 93.7 \times 2.575 \times 10^6}{6.6741 \times 10^{-11} \times 1.23 \times 10^{23}} \text{ g}$$

$$> 13.2 \text{ g}$$

4.0

1.5

Thus, all gases with atomic weight more than $A_{\min} = 13.2$ will be retained in the atmosphere of Titan.

0.5

Half mark for understanding that atomic mass has no units.

Alternative solution

$$\frac{3}{2}k_B T_T \approx \frac{1}{2}m_g v_{\text{rms}}^2$$

$$\therefore v_{\text{rms}} \approx \sqrt{\frac{3k_B T_T}{m_g}}$$

$$\therefore m_g > \frac{54k_B T_T R_T}{GM_T}$$

$$m_g > 2.19 \times 10^{-26} \text{ kg}$$

$$\therefore A_{\min} = \frac{m_g}{\text{atomic mass unit}} = \frac{2.19 \times 10^{-26} \text{ kg}}{1.66 \times 10^{-27} \text{ kg}}$$

4.0

4.0

1.0

$$A_{\min} = 13.2$$

1.0

Answers between 13.0 and 13.4 are acceptable with full credit.

(T3) **Early Universe**

Cosmological models indicate that radiation energy density, ρ_r , in the Universe is proportional to $(1+z)^4$, and the matter energy density, ρ_m , is proportional to $(1+z)^3$, where z is the redshift. The dimensionless density parameter, Ω , is given as $\Omega = \rho/\rho_c$, where ρ_c is the critical energy density of the Universe. In the present Universe, the density parameters corresponding to radiation and matter, are $\Omega_{r0} = 10^{-4}$ and $\Omega_{m0} = 0.3$, respectively.

(T3.1) Calculate the redshift, z_e , at which radiation and matter energy densities were equal.

3

Solution:

$$\frac{\rho_{m0}/\rho_c}{\rho_{r0}/\rho_c} = \frac{\Omega_{m0}}{\Omega_{r0}} = \frac{0.3}{10^{-4}} = 3000$$

At z_e , both matter density and radiation density were equal.

$$\begin{aligned}\rho_r &= \rho_m \\ \therefore \rho_{r0}(1+z_e)^4 &= \rho_{m0}(1+z_e)^3 \\ 1+z_e &= \frac{\rho_{m0}}{\rho_{r0}} = 3000 \\ \therefore z_e &\simeq \boxed{3000}\end{aligned}$$

2.0

1.0

Only $z_e = 2999$ and $z_e = 3000$ are acceptable answers.

(T3.2) Assuming that the radiation from the early Universe has a blackbody spectrum with a temperature of 2.732 K, estimate the temperature, T_e , of the radiation at redshift z_e .

4

Solution:

As the Universe behaves like an ideal black body, the radiation density will be proportional to the fourth power of the temperature (Stefan's law).

$$\begin{aligned}\left(\frac{T_e}{T_0}\right)^4 &= \frac{\rho_{re}}{\rho_{r0}} \\ &= \frac{\rho_{r0}(1+z_e)^4}{\rho_{r0}} \\ \left(\frac{T_e}{2.732}\right)^4 &= (1+z_e)^4 \\ \frac{T_e}{2.732} &= 1+z_e = 3000 \\ T_e &= 3000 \times 2.732 \\ T_e &= \boxed{8200 \text{ K}}\end{aligned}$$

2.0

1.0

1.0

$8100 \leq T_e \leq 8200$ gives 1.0; $8200 < T_e \leq 9000$ gives 0.5; else 0.

(T3.3) Estimate the typical photon energy, E_ν (in eV), of the radiation as emitted at redshift z_e .

3

Solution:

Wien's law:

$$\lambda_{\max} = \frac{0.002898 \text{ m K}}{T_e}$$

$$= \frac{0.002898}{8200} \text{ m} = 354 \text{ nm}$$

$$E_\nu = \frac{hc}{\lambda_{\max}}$$

$$= \frac{6.62 \times 10^{-34} \times 3 \times 10^8}{354 \times 10^{-9}} \text{ J} = \frac{5.62 \times 10^{-19}}{1.602 \times 10^{-19}} \text{ eV}$$

$$E_\nu = \boxed{3.5 \text{ eV}}$$

1.0

1.0

1.0

Alternative solution:

$$E_\nu = k_B T_e$$

$$= 1.38 \times 10^{-23} \times 8200 \text{ J} = \frac{1.13 \times 10^{-19}}{1.602 \times 10^{-19}} \text{ eV}$$

$$E_\nu = \boxed{0.71 \text{ eV}}$$

2.0

1.0

Use of either Wien's law or $E = k_B T$ gets full credit. $E_\nu = 3k_B T/2$ or similar gets no credit. Answers with $E_\nu = 3k_B T$ or $E_\nu = 2.7k_B T$ also get full credit.

(T4) **Shadows**

10

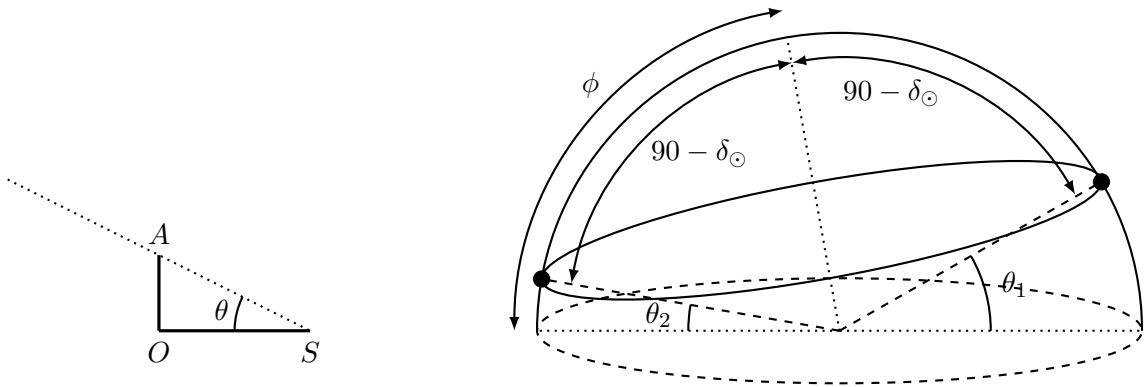
An observer in the northern hemisphere noticed that the length of the shortest shadow of a 1.000 m vertical stick on a day was 1.732 m. On the same day, the length of the longest shadow of the same vertical stick was measured to be 5.671 m.

Find the latitude, ϕ , of the observer and declination of the Sun, $\delta_\odot$, on that day. Assume the Sun to be a point source and ignore atmospheric refraction.

Solution:

As the longest shadow of the Sun on the given day is of finite length, the Sun is circumpolar for this observer on this day.

2.0



In the figure above, the left panel shows the shadow OS formed by stick OA (of length 1.000 m), and the right panel shows the Sun's location in two cases.

For an altitude θ of the Sun,

$$\tan \theta = \frac{OA}{OS} = \frac{1.000 \text{ m}}{OS}$$

$$\therefore \cot \theta = OS \text{ (in metres)}$$

1.0

Let θ_1 and θ_2 be altitude in two extreme cases.

$$\theta_1 = 180^\circ - \phi - (90^\circ - \delta_\odot) = 90^\circ - \phi + \delta_\odot$$

1.0

$$\cot(90^\circ - \phi + \delta_\odot) = 1.732$$

$$\therefore \tan(\phi - \delta_\odot) = 1.732$$

$$\phi - \delta_\odot = \tan^{-1}(1.732) = 60^\circ = 1.047 \text{ rad}$$

1.5

$$\theta_2 = \phi - (90^\circ - \delta_\odot) = \phi - 90^\circ + \delta_\odot$$

1.0

$$\cot(\phi - 90^\circ + \delta_\odot) = 5.671$$

$$\therefore \tan(\phi - 90^\circ + \delta_\odot) = \frac{1}{5.671}$$

$$\phi + \delta_\odot = \tan^{-1}\left(\frac{1}{5.671}\right) + 90^\circ = 100^\circ = 1.745 \text{ rad}$$

1.5

Solving,

$$\boxed{\phi = 80^\circ} = 1.396 \text{ rad}$$

1.0

$$\boxed{\delta_\odot = 20^\circ} = 0.349 \text{ rad}$$

1.0

Given high accuracy of shadow length, only $\pm 0.5^\circ$ is allowed.

One can also solve the question by manipulating $\tan(\phi - \delta_\odot)$ and $\tan(\phi + \delta_\odot)$, to get $\tan(\phi)$ and $\tan(\delta_\odot)$.

(T5) **GMRT beam transit**

10

Giant Metrewave Radio Telescope (GMRT), one of the world's largest radio telescopes at metre wavelengths, is located in western India (latitude: $19^\circ 6' \text{ N}$, longitude: $74^\circ 3' \text{ E}$). GMRT consists of 30 dish antennas, each with a diameter of 45.0 m. A single dish of GMRT was held fixed with its axis pointing at a zenith angle of $39^\circ 42'$ along the northern meridian such that a radio point source would pass along a diameter of the beam, when it is transiting the meridian.

What is the duration T_{transit} for which this source would be within the FWHM (full width at half maximum) of the beam of a single GMRT dish observing at 200 MHz?

Hint: The FWHM size of the beam of a radio dish operating at a given frequency corresponds to the angular resolution of the dish. Assume uniform illumination.

Solution:

As the dish is pointed towards northern meridian at zenith angle of 39.7° , altitude of the centre of the beam is

$$a = 90.00^\circ - z = 90.00^\circ - 39.70^\circ = 50.30^\circ$$

1.0

Thus, declination of the source should be,

$$\delta = 90.00 - a + \phi = 90.00^\circ - 50.30^\circ + 19.10^\circ = 58.80^\circ$$

2.0

Declination = ZA + Latitude also gets full credit.

FWHM beam size (for uniform illumination) will be given by

$$\theta = \frac{1.22\lambda}{D}$$

1.5

$$= \frac{1.22c}{D\nu} = \frac{1.22 \times 2.998 \times 10^8}{45.0 \times 2 \times 10^8}$$

$$= 0.0406 \text{ rad}$$

$$\theta = 2.33^\circ$$

1.5

$$T_{\text{transit}} = \frac{\theta \times 3.99 \text{ min}}{\cos \delta}$$

$$= \frac{2.33 \times 3.99 \text{ min}}{\cos 58.8}$$

3.0

$$T_{\text{transit}} = 17.9 \text{ min}$$

1.0

- Use of 4 min per degree is also acceptable.
- Missing $\cos \delta$ gets a penalty of 2.0.

(T6) Cepheid Pulsation

The star β -Doradus is a Cepheid variable star with a pulsation period of 9.84 days. We make a simplifying assumption that the star is brightest when it is most contracted (radius being R_1) and it is faintest when it is most expanded (radius being R_2). For simplicity, assume that the star maintains its spherical shape and behaves as a perfect black body at every instant during the entire cycle. The bolometric magnitude of the star varies from 3.46 to 4.08. From Doppler measurements, we know that during pulsation the stellar surface expands or contracts at an average radial speed of 12.8 km s^{-1} . Over the period of pulsation, the peak of thermal radiation (intrinsic) of the star varies from 531.0 nm to 649.1 nm .

(T6.1) Find the ratio of radii of the star in its most contracted and most expanded states (R_1/R_2). 7

Solution:

We first find flux ratio and then use Stefan's law to compare the fluxes.

$$m_1 - m_2 = -2.5 \log \left(\frac{F_1}{F_2} \right)$$

1.0

$$\therefore \frac{F_1}{F_2} = 10^{-0.4(m_1 - m_2)} = 10^{-0.4(3.46 - 4.08)}$$

$$= 1.77$$

1.0

$$L_i = 4\pi R_i^2 \sigma T_i^4$$

1.0

$$\therefore F_1 = \frac{4\pi R_1^2 \sigma T_1^4}{4\pi D^2}$$

$$F_2 = \frac{4\pi R_2^2 \sigma T_2^4}{4\pi D^2}$$

1.0

$$\frac{F_1}{F_2} = \frac{R_1^2}{R_2^2} \times \frac{T_1^4}{T_2^4}$$

$$\frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{T_2}{T_1}\right)^2$$

From Wien's displacement law, $\frac{T_2}{T_1} = \frac{\lambda_1}{\lambda_2}$.

$$\therefore \frac{R_1}{R_2} = \sqrt{\frac{F_1}{F_2}} \times \left(\frac{\lambda_1}{\lambda_2}\right)^2$$

$$= \sqrt{1.77} \times \left(\frac{531.0}{649.1}\right)^2$$

$$\boxed{\frac{R_1}{R_2} = 0.890}$$

Acceptable range: ± 0.010 .

1.0

1.0

1.0

- (T6.2) Find the radii of the star (in metres) in its most contracted and most expanded states (R_1 and R_2). 3

Solution:

$$R_2 - R_1 = v \times P/2$$

$$R_2 - R_1 = 12.8 \times 10^3 \times 86\,400 \times \frac{9.84}{2} \text{ m}$$

$$(1 - 0.890)R_2 = 5.441 \times 10^9 \text{ m}$$

$$\therefore \boxed{R_2 = 4.95 \times 10^{10} \text{ m}}$$

$$\boxed{R_1 = 4.41 \times 10^{10} \text{ m}}$$

Acceptable range: $\pm 0.02 \times 10^{10} \text{ m}$ for both.

2.0

0.5

0.5

- (T6.3) Calculate the flux of the star, F_2 , when it is in its most expanded state. 5

Solution:

To get the absolute value of flux (F_2) we must compare it with observed flux of the Sun.

$$m_2 - m_{\odot} = -2.5 \log \left(\frac{F_2}{F_{\odot}} \right)$$

$$\therefore F_2 = F_{\odot} 10^{-0.4(m_2 - m_{\odot})}$$

$$= \frac{L_{\odot}}{4\pi a_{\oplus}^2} \times 10^{-0.4(4.08 + 26.72)}$$

$$= \frac{3.826 \times 10^{26} \times 4.7863 \times 10^{-13}}{4\pi (1.496 \times 10^{11})^2} \text{ W m}^{-2}$$

$$\boxed{F_2 = 6.51 \times 10^{-10} \text{ W m}^{-2}}$$

Acceptable range: $\pm 0.04 \times 10^{-10} \text{ W m}^{-2}$.

3.0

2.0

(T6.4) Find the distance to the star, D_{star} , in parsecs.

5

Solution:

$$D_{\text{star}} = \sqrt{\frac{L_2}{4\pi F_2}} = \sqrt{\frac{R_2^2 \sigma T_2^4}{F_2}} = R_2 T_2^2 \sqrt{\frac{\sigma}{F_2}}$$

2.0

Wien's law: $T_2 = \frac{2.898 \times 10^{-3} \text{ m K}}{\lambda_2}$

1.0

$$D_{\text{star}} = 4.95 \times 10^{10} \times \left(\frac{2.898 \times 10^{-3}}{649.1 \times 10^{-9}} \right)^2 \sqrt{\frac{5.670 \times 10^{-8}}{6.51 \times 10^{-10}}}$$

$$\therefore D_{\text{star}} = 9.208 \times 10^{18} \text{ m} = 298 \text{ pc}$$

2.0

Acceptable range: $298 \pm 2 \text{ pc}$ (depends on truncation).

(T7) **Telescope optics**

In a particular ideal refracting telescope of focal ratio $f/5$, the focal length of the objective lens is 100 cm and that of the eyepiece is 1 cm.

(T7.1) What is the angular magnification, m_0 , of the telescope? What is the length of the telescope, L_0 , i.e. the distance between its objective and eyepiece?

4

Solution:

The magnification will be given by,

$$m_0 = \frac{f_o}{f_e} = \frac{100}{1} = 100$$

1.0

1.0

The magnification is $m_0 = 100$

Length of the telescope will be

$$L_0 = f_o + f_e = 100 + 1 = 101 \text{ cm}$$

1.0

1.0

The telescope length will be $L_0 = 101 \text{ cm}$

Exact answer required for credit.

An introduction of a concave lens (Barlow lens) between the objective lens and the prime focus is a common way to increase the magnification without a large increase in the length of the telescope. A Barlow lens of focal length 1 cm is now introduced between the objective and the eyepiece to double the magnification.

(T7.2) At what distance, d_B , from the prime focus must the Barlow lens be kept in order to obtain this desired double magnification?

6

Solution:

We use the following sign convention. Lens is the origin. Direction along the direction

of light is taken as positive. The lens formula is $\frac{1}{f} = \frac{1}{v} - \frac{1}{u}$ (f is positive for convex lens and negative for concave lens). Magnification is $m = \frac{v}{u}$. Solutions using other sign conventions are acceptable.

Let v be image distance from the Barlow lens.

$$\frac{1}{f_B} = \frac{1}{v} - \frac{1}{u}$$

1.0

Distance of Barlow lens, d_B , before the prime focus is same as the object distance, u , in this case.

1.0

$$\frac{1}{f_B} = \frac{1}{v} - \frac{1}{d_B}$$

$$\text{Also, } m_B = 2 = \frac{v}{u} = \frac{v}{d_B}$$

0.5

$$\therefore \frac{1}{d_B} = \frac{2}{v}$$

1.0

$$\therefore \frac{1}{-1} = \frac{1}{v} - \frac{2}{v}$$

$$-1 = \frac{-1}{v}$$

$$v = 1 \text{ cm}$$

$$d_B = \frac{v}{2} = \frac{1 \text{ cm}}{2} = 0.5 \text{ cm}$$

2.5

The positive sign for d_B indicates that the Barlow lens was introduced 0.5 cm before the prime focus.

(T7.3) What is the increase, ΔL , in the length of the telescope?

4

Solution:

The increase in the length will be,

$$\Delta L = v - d_B$$

2.0

$$= 1.0 - 0.5 = 0.5 \text{ cm}$$

2.0

Thus, the length will be increased by $\Delta L = 0.5 \text{ cm}$

A telescope is now constructed with the same objective lens and a CCD detector placed at the prime focus (without any Barlow lens or eyepiece). The size of each pixel of the CCD detector is $10 \mu\text{m}$.

(T7.4) What will be the distance in pixels between the centroids of the images of the two stars, n_p , on the CCD, if they are $20''$ apart on the sky?

6

Solution:

Plate scale at prime focus is given by,

$$s = \frac{1}{f_o} = \frac{1 \text{ rad}}{1 \text{ m}} = 0.206 \, 265 \text{ arcsec}/\mu\text{m}$$

2.0

Since each pixel is $10\ \mu\text{m}$ in size,

$$s_p = 10 \times 0.206\ \text{arcsec}/\mu\text{m} = 2.06\ \text{arcsec}/\text{pixel}$$

Two stars will be separated by,

$$n_p = \frac{20''}{2.06''}\ \text{pixels} \simeq \boxed{10\ \text{pixels}}$$

Acceptable range: 9.5 to 10.5 pixels.

2.0

2.0

(T8) **U-band photometry**

A star has an apparent magnitude $m_U = 15.0$ in the U -band. The U -band filter is ideal, i.e., it has perfect (100%) transmission within the band and is completely opaque (0% transmission) outside the band. The filter is centered at $360\ \text{nm}$, and has a width of $80\ \text{nm}$. It is assumed that the star also has a flat energy spectrum with respect to frequency. The conversion between magnitude, m , in any band and flux density, f , of a star in Jansky ($1\ \text{Jy} = 1 \times 10^{-26}\ \text{W Hz}^{-1}\ \text{m}^{-2}$) is given by

$$f = 3631 \times 10^{-0.4m}\ \text{Jy}$$

(T8.1) Approximately how many U -band photons, N_0 , from this star will be incident normally on a $1\ \text{m}^2$ area at the top of the Earth's atmosphere every second?

8

Solution:

The U -band is defined as $(360 \pm 40)\ \text{nm}$. Thus, the maximum, minimum and average frequencies of the band will be,

$$\nu_{\max} = \frac{c}{\lambda_{\max}} = 9.369 \times 10^{14}\ \text{Hz}$$

$$\nu_{\min} = 7.495 \times 10^{14}\ \text{Hz}$$

$$\nu_{\text{avg}} = 8.432 \times 10^{14}\ \text{Hz}$$

$$\Delta\nu = \nu_{\max} - \nu_{\min}$$

$$= 1.874 \times 10^{14}\ \text{Hz}$$

$$f_{\text{st1}} = 3631 \times 10^{-0.4 \times 15}$$

$$= 3.631\ \text{mJy} = 3.631 \times 10^{-29}\ \text{W Hz}^{-1}\ \text{m}^{-2}$$

$$\text{Now, } N_0 \times h\nu_{\text{avg}} = \Delta\nu \times f_{\text{st1}} \times A \times \Delta t$$

$$\text{where, } A = 1\ \text{m}^2 \ \& \ \Delta t = 1\ \text{s}$$

$$\therefore N_0 = \frac{1.874 \times 10^{14} \times 3.631 \times 10^{-29}}{6.626 \times 10^{-34} \times 8.432 \times 10^{14}}$$

$$\simeq \boxed{12180}$$

2.0

2.0

2.0

2.0

Exact calculation including integration is accepted with full credit (exact answer: 12190).

Accepted range: 12180 ± 200 .

Using flat spectrum for $\Delta\lambda$ instead of $\Delta\nu$ is considered a major conceptual error, and will incur penalty of 2.0 marks.

This star is being observed in the U -band using a ground based telescope, whose primary mirror has a diameter of $2.0\ \text{m}$. Atmospheric extinction in U -band during the observation is 50%. You may assume that the seeing is diffraction limited. Average surface brightness of night sky in U -band was measured to be $22.0\ \text{mag}/\text{arcsec}^2$.

- (T8.2) What is the ratio, R , of number of photons received per second from the star to that received from the sky, when measured over a circular aperture of diameter $2''$? 8

Solution:

Let us call sky flux per square arcsec as Φ and total sky flux for the given aperture as ϕ_{sky} . Let total star flux be ϕ_{st} .

$$\phi_{\text{sky}} = A\Phi = \pi \times (1 \text{ arcsec})^2 \times \Phi = \pi\Phi$$

$$\therefore m_{\text{sky}} = 22.0 + 2.5 \log_{10} \left(\frac{\Phi}{\phi_{\text{sky}}} \right)$$

$$= 22.0 + 2.5 \log_{10} \left(\frac{\Phi}{\pi\Phi} \right)$$

$$= 22.0 - 2.5 \log_{10} (\pi)$$

$$m_{\text{sky}} = 20.76 \text{ mag}$$

As extinction is 50%

$$R = \frac{\phi_{\text{st}2}}{\phi_{\text{sky}}} = \frac{0.5\phi_{\text{st}1}}{\phi_{\text{sky}}} = 0.5 \times 10^{(20.76-15)/2.5}$$

$$\simeq \boxed{100}$$

Accepted range: ± 5 .

A student may also calculate number of photons incident per second per metre square in this case and then compare it with the answer in the first case to get the correct ratio.

- (T8.3) In practice, only 20% of U -band photons falling on the primary mirror are detected. How many photons, N_t , from the star are detected per second? 4

Solution:

$$N_t \times 1 \text{ m}^2 = N_0 \times 0.5 \times 0.2 \times A_t$$

$$N_t = 12180 \times 0.5 \times 0.2 \times \pi \left(\frac{2.0}{2} \right)^2 = 1233\pi$$

$$N_t \simeq \boxed{3813}$$

Accepted range (3813 ± 50) .

(T9) Mars Orbiter Mission

India's Mars Orbiter Mission (MOM) was launched using the Polar Satellite Launch Vehicle (PSLV) on 5 November 2013. The dry mass of MOM (body + instruments) was 500 kg and it carried fuel of mass 852 kg. It was initially placed in an elliptical orbit around the Earth with perigee at a height of 264 km and apogee at a height of 23 904 km, above the surface of the Earth. After raising the orbit six times, MOM was transferred to a trans-Mars injection orbit (Hohmann orbit).

The first such orbit-raising was performed by firing the engines for a very short time near the perigee. The engines were fired to change the orbit without changing the plane of the orbit and without changing its perigee. This gave a net impulse of $1.73 \times 10^5 \text{ kg m s}^{-1}$ to the satellite. Ignore the change in mass due to burning of fuel.

- (T9.1) What is the height of the new apogee, h_a , above the surface of the Earth, after this 14

engine burn?

Solution:

Let apogee and perigee distances be r_a and r_p respectively.

$$r_p = R_{\oplus} + h_p^i = (6371 + 264) \text{ km} = 6635 \text{ km} \quad 1.0$$

$$r_a = R_{\oplus} + h_a^i = (6371 + 23904) \text{ km} = 30275 \text{ km} \quad 1.0$$

Conservation of energy and angular momentum gives

$$E = -\frac{GmM_{\oplus}}{r_p + r_a}$$

Total energy at perigee

$$\frac{1}{2}mv_p^2 - \frac{GmM_{\oplus}}{r_p} = E = -\frac{GmM_{\oplus}}{r_p + r_a}$$

$$\frac{1}{2}v_p^2 = \frac{GM_{\oplus}}{r_p} \left(1 - \frac{r_p}{r_p + r_a}\right)$$

$$\therefore v_p = \sqrt{2 \frac{GM_{\oplus}}{r_a + r_p} \frac{r_a}{r_p}} \quad 2.0$$

$$= \sqrt{\frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24} \times 3.0275 \times 10^7}{6.635 \times 10^6 \times (3.0275 + 6.635 \times 10^6)}}$$

$$= 9.929 \text{ km s}^{-1} \quad 1.0$$

As the engine burn is just 41.6 s, we assume that the entire impulse is applied instantaneously at perigee. The impulse is $J = 1.73 \times 10^5 \text{ kg m s}^{-1}$. Note that the total mass of MOM must include the fuel, so we have to use

$$m = 500 + 852 = 1352 \text{ kg.} \quad 1.0$$

Change in velocity due to impulse at perigee is

$$\Delta v = \frac{J}{m} = \frac{1.73 \times 10^5}{1352} = 128.0 \text{ m s}^{-1} \quad 1.0$$

The new velocity will be given by (we use ' symbol to denote quantities after the first orbit-raising manoeuvre)

$$v'_p = v_p + \Delta v = 10.06 \text{ km s}^{-1} \quad 1.0$$

The perigee remains unchanged. So we get $r'_p = r_p$.

Since the satellite is moving faster, the new apogee will be higher.

$$v'_p = \sqrt{2GM_{\oplus} \frac{r'_a}{r'_p(r'_a + r'_p)}} \quad 2.0$$

$$\therefore 1 + \frac{r'_p}{r'_a} = \frac{2GM_{\oplus}}{(v'_p)^2 \times r'_p}$$

$$= \frac{2 \times 6.674 \times 10^{-11} \times 5.972 \times 10^{24}}{(10.06 \times 10^3)^2 \times 6.635 \times 10^6} = 1.188$$

$$r'_a = \frac{6635}{0.188} = 35380 \text{ km} \quad 2.0$$

$$h_a = 35380 - 6371$$

$$= \boxed{29009 \text{ km}} \quad 1.0$$

Acceptable range: ± 150 km

(T9.2) Find eccentricity (e) of the new orbit after the burn and new orbital period (P) of MOM in hours.

6

Solution:

As seen above,

$$\frac{r'_p}{r'_a} = 0.188 = \frac{1 - e}{1 + e}$$

1.0

$$\therefore e = \frac{1 - 0.188}{1 + 0.188} = \boxed{0.683}$$

1.0

Acceptable range: ± 0.002

The new orbital semi-major axis and orbital period will be,

$$a' = \frac{r'_a + r'_p}{2} = \frac{35380 + 6635}{2} = 20933 \text{ km}$$

1.0

1.0

$$P = 2\pi \sqrt{\frac{a'^3}{GM_\oplus}}$$

1.0

$$= 2\pi \sqrt{\frac{(2.0933 \times 10^7)^3}{6.674 \times 10^{-11} \times 5.972 \times 10^{24}}} = \boxed{30136 \text{ s} = 8.37 \text{ h}}$$

1.0

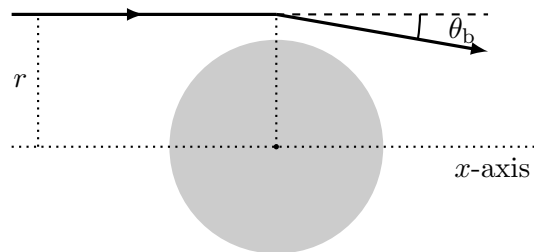
Acceptable range: ± 0.1 h

(T10) **Gravitational Lensing Telescope**

Einstein's General Theory of Relativity predicts bending of light around massive bodies. For simplicity, we assume that the bending of light happens at a single point for each light ray, as shown in the figure. The angle of bending, θ_b , is given by

$$\theta_b = \frac{2R_{\text{sch}}}{r}$$

where R_{sch} is the Schwarzschild radius associated with that gravitational body. We call r , the distance of the incoming light ray from the parallel x -axis passing through the centre of the body, as the "impact parameter".



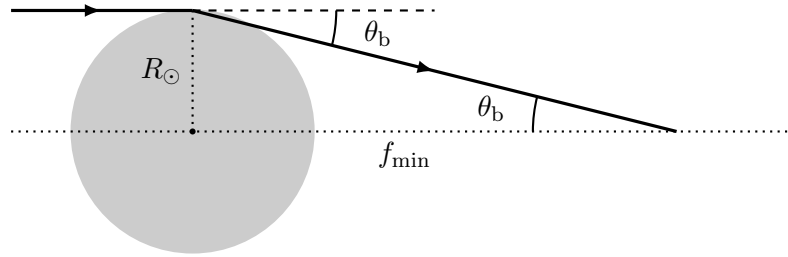
A massive body thus behaves somewhat like a focusing lens. The light rays coming from infinite distance beyond a massive body, and having the same impact parameter r , converge at a point along the axis, at a distance f_r from the centre of the massive body. An observer at that point

will benefit from huge amplification due to this gravitational focusing. The massive body in this case is being used as a Gravitational Lensing Telescope for amplification of distant signals.

- (T10.1) Consider the possibility of our Sun as a gravitational lensing telescope. Calculate the shortest distance, $f_{\min}$, from the centre of the Sun (in A.U.) at which the light rays can get focused. 6

Solution:

The rays travelling closer to the gravitational body will bend more. Thus, we get shortest convergence point where the rays just grazing the solar surface will meet each other.



$$\begin{aligned}\theta_b &= \frac{2R_{\text{sch}}}{R_{\odot}} \simeq \frac{R_{\odot}}{f_{\min}} \\ \therefore f_{\min} &= \frac{R_{\odot}^2}{2R_{\text{sch}}} = \frac{R_{\odot}^2 c^2}{4GM_{\odot}} \\ &= \frac{(6.955 \times 10^8 \times 2.998 \times 10^8)^2}{4 \times 6.674 \times 10^{-11} \times 1.989 \times 10^{30}} \text{ m} \\ &= 8.188 \times 10^{13} \text{ m} = \frac{8.188 \times 10^{13}}{1.496 \times 10^{11}} \text{ AU}\end{aligned}$$

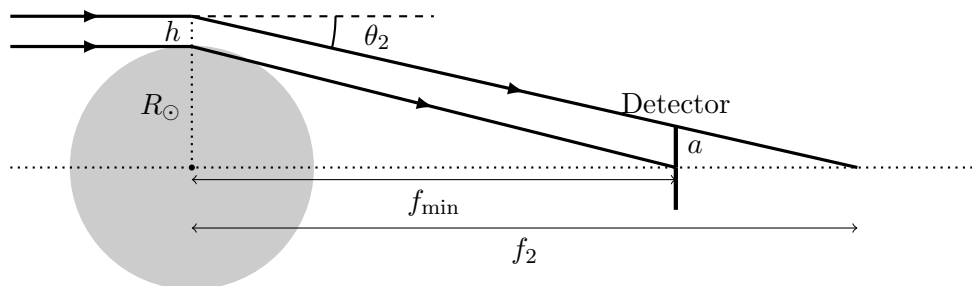
$$f_{\min} = 547.3 \text{ AU}$$

- (T10.2) Consider a small circular detector of radius a , kept at a distance $f_{\min}$ centred on the x -axis and perpendicular to it. Note that only the light rays which pass within a certain annulus (ring) of width h (where $h \ll R_{\odot}$) around the Sun would encounter the detector. The amplification factor at the detector is defined as the ratio of the intensity of the light incident on the detector in the presence of the Sun and the intensity in the absence of the Sun. 8

Express the amplification factor, A_m , at the detector in terms of $R_{\odot}$ and a .

Solution:

The following figure needs to be drawn. 1.0



The light bending from the surface of the Sun ($r = R_\odot$) will intersect the detector at its centre, as it is kept at $f_{\min}$.

The boundary of the detector will be intersected by a light ray with $r = R_\odot + h$. This ray will intersect the x -axis at a distance f_2 .

$$f_2 = \frac{(R_\odot + h)^2}{2R_{\text{sch}}}$$

2.0

Same argument as in Part 1

For small angles,

$$\begin{aligned} a &= (f_2 - f_{\min})\theta_2 \\ &= \left[\frac{(R_\odot + h)^2}{2R_{\text{sch}}} - \frac{R_\odot^2}{2R_{\text{sch}}} \right] \frac{2R_{\text{sch}}}{(R_\odot + h)} = \frac{2R_\odot h + h^2}{R_\odot + h} \\ &\simeq 2h \end{aligned}$$

2.0

Let original intensity of the incoming radiation be I_0 .

The flux at the detector in the presence of Sun is $\Phi_\odot = I_0 2\pi R_\odot h$

1.0

The flux at the detector in the absence of Sun is $\Phi_0 = I_0 \pi a^2$

1.0

The amplification is therefore

$$A_m = \frac{\Phi_\odot}{\Phi_0} = \frac{I_0 2\pi R_\odot h}{I_0 \pi a^2} = \left[\frac{R_\odot}{a} \right]$$

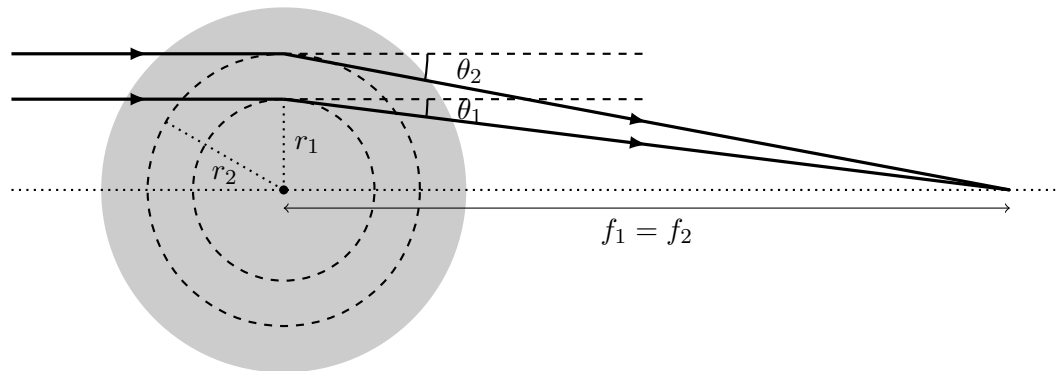
1.0

(T10.3) Consider a spherical mass distribution, such as a dark matter cluster, through which light rays can pass while undergoing gravitational bending. Assume for simplicity that for the gravitational bending with impact parameter, r , only the mass $M(r)$ enclosed inside the radius r is relevant.

6

What should be the mass distribution, $M(r)$, such that the gravitational lens behaves like an ideal optical convex lens ?

Solution:



All rays should focus at the same spot. This should be evident from figure drawn on answersheet or otherwise.

2.0

Let there be two rays with impact parameters r_1 and r_2 . The corresponding distances of focus will be

$$f_i = \frac{r_i^2}{2r_{\text{sch}_i}} = \frac{r_i^2 c^2}{4GM(r_i)}$$

2.0

Same argument as in Part 1

The requirement $f_1 = f_2$ implies

$$\frac{r_1^2}{r_2^2} = \frac{M(r_1)}{M(r_2)}$$

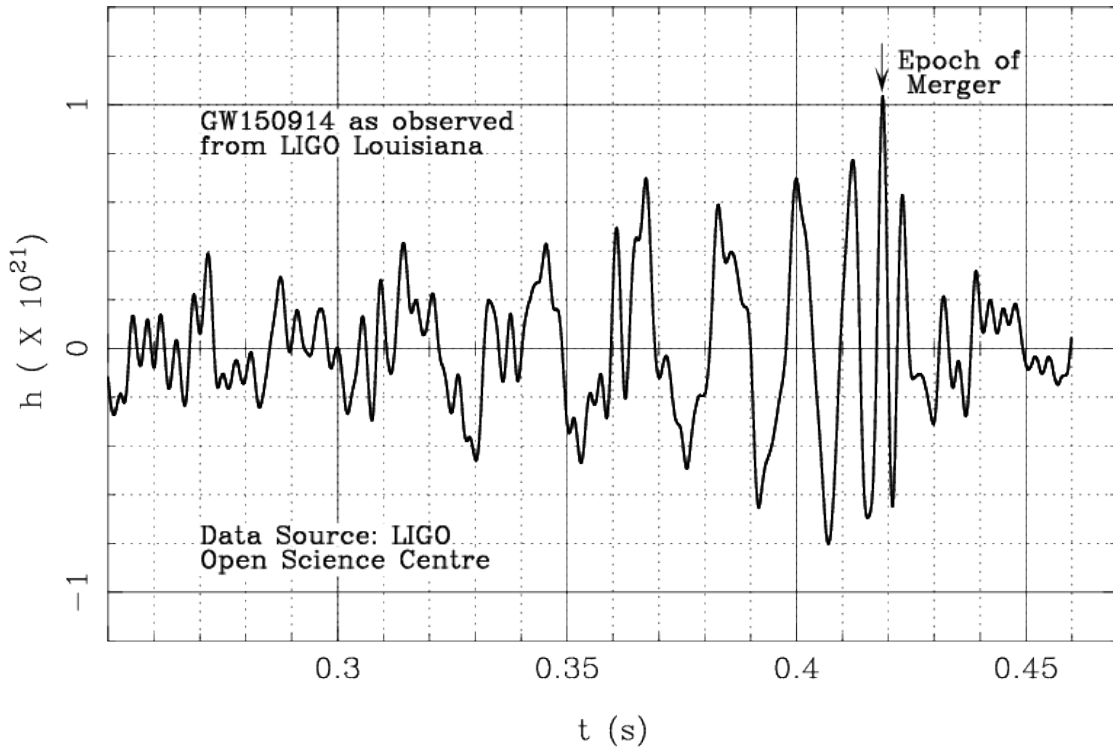
1.0

The required mass distribution is: $M(r) \propto r^2$

1.0

(T11) **Gravitational Waves**

The first signal of gravitational waves was observed by two advanced LIGO detectors at Hanford and Livingston, USA in September 2015. One of these measurements (strain vs time in seconds) is shown in the accompanying figure. In this problem, we will interpret this signal in terms of a small test mass m orbiting around a large mass M (i.e., $m \ll M$), by considering several models for the nature of the central mass.



The test mass loses energy due to the emission of gravitational waves. As a result the orbit keeps on shrinking, until the test mass reaches the surface of the object, or in the case of a black hole, the innermost stable circular orbit – ISCO – which is given by $R_{\text{ISCO}} = 3R_{\text{sch}}$, where R_{sch} is the Schwarzschild radius of the black hole. This is the “epoch of merger”. At this point, the amplitude of the gravitational wave is maximum, and so is its frequency, which is always twice the orbital frequency. In this problem, we will only focus on the gravitational waves before the merger, when Kepler’s laws are assumed to be valid. After the merger, the form of gravitational waves will drastically change.

(T11.1) Consider the observed gravitational waves shown in the figure above. Estimate the time period, T_0 , and hence calculate the frequency, f_0 , of gravitational waves just before the epoch of merger.

Solution:

From the graph, just before the peak of emission, the time period of gravitational waves is approximately (0.007 ± 0.004) s.

$$T_0 \approx \boxed{0.007 \text{ s}}$$

2.0

Acceptable range: 0.003 to 0.011 s

That is, the frequency of the gravitational waves is $f_0 \approx \boxed{142.86 \text{ Hz}}$.

1.0

Acceptable range: 333.33 to 90.91 Hz

Answer given in terms of angular frequency with correct value and units gains full credit.

- (T11.2) For any main sequence (MS) star, the radius of the star, R_{MS} , and its mass, M_{MS} , are related by a power law given as,

10

$$R_{\text{MS}} \propto (M_{\text{MS}})^\alpha$$

$$\text{where } \alpha = 0.8 \quad \text{for } M_\odot < M_{\text{MS}}$$

$$= 1.0 \quad \text{for } 0.08M_\odot \leq M_{\text{MS}} \leq M_\odot$$

If the central object were a main sequence star, write an expression for the maximum frequency of gravitational waves, f_{MS} , in terms of mass of the star in units of solar masses ($M_{\text{MS}}/M_\odot$) and α .

Solution:

Since $m \ll M$ then, by Kepler's third law

$$f_{\text{orbital}} = \frac{1}{2\pi} \sqrt{\frac{GM}{r^3}}$$

4.0

Hence the frequency of the gravitational waves is

$$f_{\text{grav}} = 2f_{\text{orbital}} = \frac{1}{\pi} \sqrt{\frac{GM}{r^3}}$$

1.0

The frequency will be maximum when $r = R_{\text{MS}}$.

1.0

For main sequence stars,

$$\frac{R_{\text{MS}}}{R_\odot} = \left(\frac{M_{\text{MS}}}{M_\odot} \right)^\alpha$$

$$\therefore R_{\text{MS}} = R_\odot \left(\frac{M_{\text{MS}}}{M_\odot} \right)^\alpha$$

1.0

$$\begin{aligned} \therefore f_{\text{MS}} &= \frac{1}{\pi} \sqrt{\frac{GM_{\text{MS}}}{R_{\text{MS}}^3}} \left(\frac{M_\odot}{M_{\text{MS}}} \right)^{3\alpha/2} \\ &= \frac{1}{\pi} \sqrt{\frac{GM_\odot}{R_\odot^3}} \left(\frac{M_\odot}{M_{\text{MS}}} \right)^{(3\alpha-1)/2} \end{aligned}$$

$$\boxed{f_{\text{MS}} = \frac{1}{\pi} \sqrt{\frac{GM_\odot}{R_\odot^3}} \left(\frac{M_{\text{MS}}}{M_\odot} \right)^{(1-3\alpha)/2}}$$

3.0

- (T11.3) Using the above result, determine the appropriate value of α that will give the maximum possible frequency of gravitational waves, $f_{\text{MS,max}}$ for any main sequence star. Evaluate

9

this frequency.

Solution:

For possible values of α given in the question, the exponent $\frac{1-3\alpha}{2}$ is negative. Thus, if $M_{\text{MS}} > M_{\odot}$, the frequency will be smaller. Thus, for highest possible frequency coming from a main sequence star, you should take lowest possible mass i.e. $\alpha = 1.0$

$$\begin{aligned} f_{\text{MS,max}} &= \frac{1}{\pi} \sqrt{\frac{GM_{\odot}}{R_{\odot}^3}} \left(\frac{M_{\text{MS}}}{M_{\odot}} \right)^{\left(\frac{1-3 \times 1}{2}\right)} \\ &= \frac{1}{\pi} \sqrt{\frac{GM_{\odot}}{R_{\odot}^3}} \times \frac{M_{\odot}}{M_{\text{MS}}} \end{aligned}$$

The frequency of gravitational waves will be given by,

$$f_{\text{MS,max}} = \frac{1}{\pi} \sqrt{\frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{(6.955 \times 10^8)^3}} \times \frac{1}{0.08}$$

$$f_{\text{MS,max}} = 2.5 \text{ mHz}$$

Answer given in terms of angular frequency with correct value and units gains full credit.

- (T11.4) White dwarf (WD) stars have a maximum mass of $1.44M_{\odot}$ (known as the Chandrasekhar limit) and obey the mass-radius relation $R \propto M^{-1/3}$. The radius of a solar mass white dwarf is equal to 6000 km. Find the highest frequency of emitted gravitational waves, $f_{\text{WD,max}}$, if the test mass is orbiting a white dwarf.

Solution:

The maximum frequency would be when $r = R_{\text{WD}}$.

We use the notation $R_{\text{WD}\odot}$ for the radius of a solar mass white dwarf. Then for white dwarfs

$$\begin{aligned} R_{\text{WD}}^3 &= R_{\text{WD}\odot}^3 \frac{M_{\text{WD}}}{M_{\odot}} \\ f_{\text{WD}} &= \frac{1}{\pi} \sqrt{\frac{GM_{\text{WD}}}{R_{\text{WD}}^3}} \\ &= \frac{1}{\pi} \sqrt{\frac{GM_{\odot}}{R_{\text{WD}\odot}^3}} \frac{M_{\text{WD}}}{M_{\odot}} \end{aligned}$$

For maximum frequency, we have to take highest white dwarf mass.

$$\begin{aligned} f_{\text{WD,max}} &= 2.600 \times 10^{-6} \times \sqrt{\frac{1.989 \times 10^{30}}{(6000 \times 10^3)^3}} \times 1.44 \\ &= 2.600 \times 10^{-6} \times 95.96 \times 10^3 \times 1.44 \end{aligned}$$

$$f_{\text{WD,max}} = 0.359 \text{ Hz}$$

Answer given in terms of angular frequency with correct value and units gains full credit.

- (T11.5) Neutron stars (NS) are a peculiar type of compact objects which have masses between

1 and $3M_{\odot}$ and radii in the range 10–15 km. Find the range of frequencies of emitted gravitational waves, $f_{\text{NS,min}}$ and $f_{\text{NS,max}}$, if the test mass is orbiting a neutron star at a distance close to the neutron star radius.

Solution:

$$f_{\text{grav}} = \frac{1}{\pi} \sqrt{\frac{GM_{\text{NS}}}{R_{\text{NS}}^3}}$$

The lowest possible frequency is when M_{NS} is lowest and R_{NS} is the highest.

For $M_{\text{NS}} = M_{\odot}$ and $R = 15$ km, we get

$$f_{\text{NS,min}} = 1.996 \text{ kHz}$$

Similarly, the largest possible frequency is when M_{NS} is the largest and R_{NS} is the smallest.

For $M_{\text{NS}} = 3M_{\odot}$ and $R = 10$ km, we get

$$f_{\text{NS,max}} = 6.352 \text{ kHz}$$

Answer given in terms of angular frequency with correct value and units gains full credit.

- (T11.6) If the test mass is orbiting a black hole (BH), write the expression for the frequency of emitted gravitational waves, f_{BH} , in terms of mass of the black hole, M_{BH} , and the solar mass $M_{\odot}$.

Solution:

For black holes, we have to consider R_{ISCO} .

Hence the equation will be,

$$f_{\text{BH}} = \frac{1}{\pi} \times \sqrt{\frac{GM_{\odot}}{27R_{\text{sch-}\odot}^3}} \times \frac{M_{\odot}}{M_{\text{BH}}}$$

$$f_{\text{BH}} = 4.396 \text{ kHz} \times \frac{M_{\odot}}{M_{\text{BH}}}$$

Answer given in terms of angular frequency with correct value and units gains full credit.

- (T11.7) Based only on the time period (or frequency) of gravitational waves before the epoch of merger, determine whether the central object can be a main sequence star (MS), a white dwarf (WD), a neutron star (NS), or a black hole (BH). Tick the correct option in the Summary Answersheet. Estimate the mass of this object, M_{obj} , in units of $M_{\odot}$.

Solution:

We found the frequency of the LIGO-detected wave to be 166.67 Hz just before merger. As per our analysis above, only black holes can lead to emission in this frequency range.

Black Hole

By using corresponding expression,

$$M_{\text{obj}} = \frac{4396}{142.86} M_{\odot} \approx 31 M_{\odot}$$

Any answer between 13 to 50 will get full credit.

(T12) **Exoplanets**

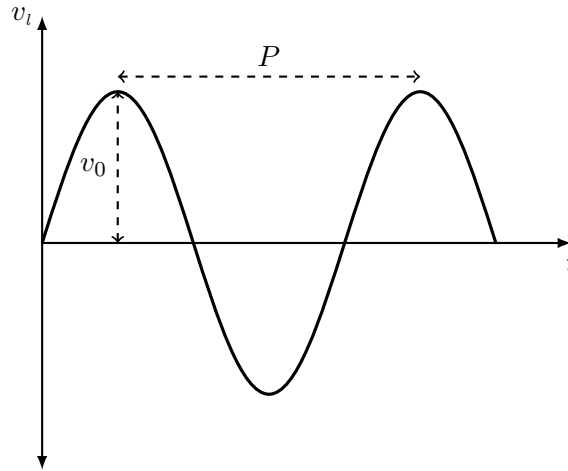
Two major methods of detection of exoplanets (planets around stars other than the Sun) are the radial velocity (or so-called “wobble”) method and the transit method. In this problem, we find out how a combination of the results of these two methods can reveal a lot of information about an orbiting exoplanet and its host star.

Throughout this problem, we consider the case of a planet of mass M_p and radius R_p moving in a circular orbit of radius a around a star of mass M_s ($M_s \gg M_p$) and radius R_s . The normal to the orbital plane of the planet is inclined at angle i with respect to the line of sight ($i = 90^\circ$ would mean “edge on” orbit). We assume that there is no other planet orbiting the star and $R_s \ll a$.

“Wobble” Method:

When a planet and a star orbit each other around their barycentre, the star is seen to move slightly, or “wobble”, since the centre of mass of the star is not coincident with the barycentre of the star-planet system. As a result, the light received from the star undergoes a small Doppler shift related to the velocity of this wobble.

The line of sight velocity, v_l , of the star can be determined from the Doppler shift of a known spectral line, and its periodic variation with time, t , is shown in the schematic diagram below. In the diagram, the two measurable quantities in this method, namely, the orbital period P and maximum line of sight velocity v_0 are shown.



(T12.1) Derive expressions for the orbital radius (a) and orbital speed (v_p) of the planet in terms of M_s and P .

3

Solution:

Kepler’s law:

$$a = \left(\frac{GM_s}{4\pi^2} P^2 \right)^{1/3}$$

1.0

Gravitational force provides centripetal acceleration:

$$v_p = \sqrt{\frac{GM_s}{a}}$$

0.5

$$v_p = \left(\frac{2\pi GM_s}{P} \right)^{1/3}$$

1.5

(T12.2) Obtain a lower limit on the mass of the planet, $M_{p,\min}$ in terms of M_s , v_0 and v_p .

4

Solution:

Momentum conservation:

$$M_p v_p = M_s v_s$$

Observed quantity is $v_0 = v_s \sin i$. Thus,

$$M_{p,\min} = M_p \sin i = \frac{M_s v_s \sin i}{v_p} = \boxed{\frac{M_s v_0}{v_p}}$$

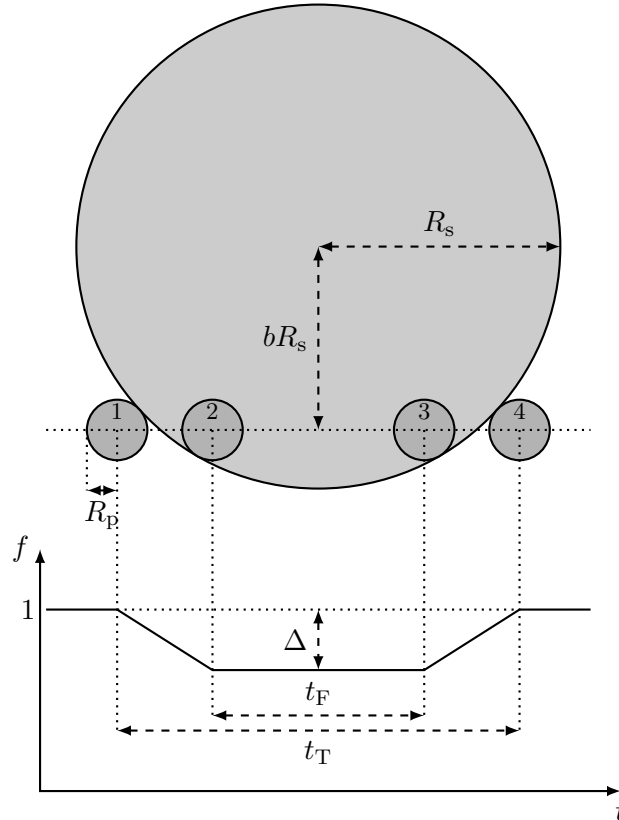
This is a lower limit on M_p .

1.5

2.5

Transit Method:

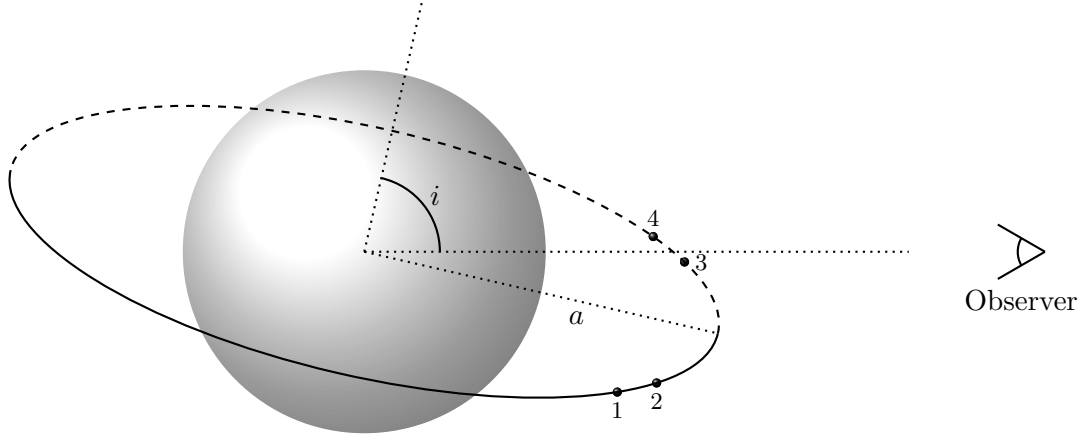
As a planet orbits its host star, for orientations of the orbital plane that are close to “edge-on” ($i \approx 90^\circ$), it will pass periodically, or “transit”, in front of the stellar disc as seen by the observer. This would cause a tiny decrease in the observed stellar flux which can be measured. The schematic diagram below (NOT drawn to scale) shows the situation from the observer’s perspective and the resulting transit light curve (normalised flux, f , vs time, t) for a uniformly bright stellar disc.



If the inclination angle i is exactly 90° , the planet would be seen to cross the stellar disc along a diameter. For other values of i , the transit occurs along a chord, whose centre lies at a distance bR_s from the centre of the stellar disc, as shown. The no-transit flux is normalised to 1 and the maximum dip during the transit is given by Δ .

The four significant points in the transit are the first, second, third and fourth contacts, marked by the positions 1 to 4, respectively, in the figure above. The time interval during the second and third contacts is denoted by t_F , when the disc of the planet overlaps the stellar disc fully. The time interval between the first and fourth contacts is denoted by t_T .

These points are also marked in the schematic diagram below showing a “side-on” view of the orbit (NOT drawn to scale).



The measurable quantities in the transit method are P , t_T , t_F and Δ .

(T12.3) Find the constraint on i in terms of R_s and a for the transit to be visible at all to the distant observer. 2

Solution:

$$bR_s = a \cos i$$

1.0

Therefore, for visibility, $0 \leq b \leq 1 \Rightarrow i \geq \cos^{-1}(R_s/a)$

1.0

(T12.4) Express Δ in terms of R_s and R_p . 1

Solution:

Blackbody $\Rightarrow$ brightness is proportional to area. Since the observer is far away from the star-planet system, size of silhouette of planet on stellar disc is independent of a .

$$\Delta = \left(\frac{R_p}{R_s} \right)^2$$

1.0

(T12.5) Express t_T and t_F in terms of R_s , R_p , a , P and b . 8

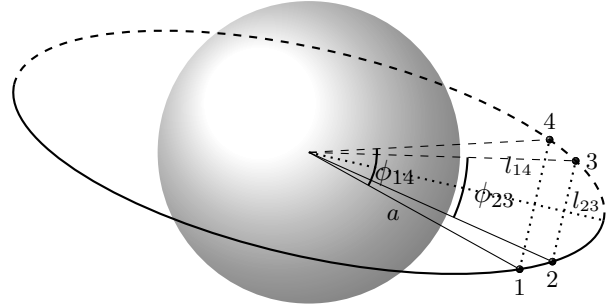
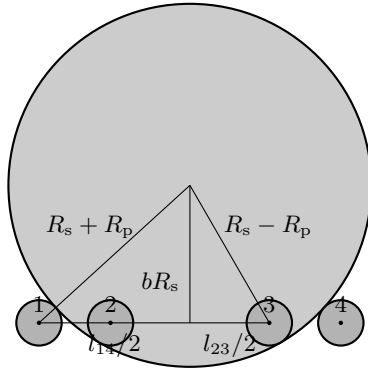
Solution:

Circular orbit $\Rightarrow$ uniform orbital speed

$$\Rightarrow \frac{t}{P} = \frac{a\phi}{2\pi a} = \frac{\phi}{2\pi}$$

1.0

where ϕ is the angle subtended by the planet at the centre of the star during transit (over time t).



$$(l_{23}/2)^2 = (R_s - R_p)^2 - (bR_s)^2$$

$$l_{23} = 2R_s \sqrt{(1 - R_p/R_s)^2 - b^2}$$

$$\sin(\phi_{23}/2) = \frac{l_{23}/2}{a}$$

$$\Rightarrow \phi_{23} = 2 \sin^{-1} \left(\frac{l_{23}}{2a} \right) = 2 \sin^{-1} \left[\frac{R_s}{a} \sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

$$t_F = \frac{P}{2\pi} \phi_{23} = \frac{P}{\pi} \sin^{-1} \left[\frac{R_s}{a} \sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

Similarly,

$$t_T = \frac{P}{\pi} \sin^{-1} \left[\frac{R_s}{a} \sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

(T12.6) In the approximation of an orbit much larger than the stellar radius, show that the parameter b is given by

$$b = \left[1 + \Delta - 2\sqrt{\Delta} \frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2} \right]^{1/2}$$

Solution:

Since $R_s \ll a$, use $\sin^{-1} x \approx x$.

$$t_T \approx \frac{P}{\pi} \left[\frac{R_s}{a} \sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

$$t_F \approx \frac{P}{\pi} \left[\frac{R_s}{a} \sqrt{\left(1 - \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

2.0

2.0

1.0

1.0

1.0

5

2.0

Dividing, and putting $R_p/R_s = \sqrt{\Delta}$,

$$\frac{t_F}{t_T} = \left[\frac{(1 - \sqrt{\Delta})^2 - b^2}{(1 + \sqrt{\Delta})^2 - b^2} \right]^{1/2}$$

1.0

$$\Rightarrow b = \left[\frac{(1 - \sqrt{\Delta})^2 - \left(\frac{t_F}{t_T}\right)^2 (1 + \sqrt{\Delta})^2}{1 - \left(\frac{t_F}{t_T}\right)^2} \right]^{1/2} = \left[\frac{1 + \Delta - 2\sqrt{\Delta} \frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2}}{1 - \left(\frac{t_F}{t_T}\right)^2} \right]^{1/2}$$

2.0

Expressions lacking the use of approximation $R_s/a \ll 1$, but otherwise correct will get a penalty of 2.0.

Use of approximation with proper justification at a later stage than at the first step will get full credit.

(T12.7) Use the result of part (T12.6) to obtain an expression for the ratio a/R_s in terms of measurable transit parameters, using a suitable approximation. 3

Solution:

$$t_T = \frac{P}{\pi} \left[\frac{R_s}{a} \sqrt{\left(1 + \frac{R_p}{R_s}\right)^2 - b^2} \right]$$

1.0

Either substitution of b or elimination of b gets 1.0.

Substituting b and R_p/R_s ,

$$t_T = \frac{P}{\pi} \left[\frac{R_s}{a} \sqrt{(1 + \sqrt{\Delta})^2 - 1 - \Delta + 2\sqrt{\Delta} \frac{1 + \left(\frac{t_F}{t_T}\right)^2}{1 - \left(\frac{t_F}{t_T}\right)^2}} \right]$$

$$\Rightarrow t_T = \frac{P R_s}{\pi a} \left[\frac{4\sqrt{\Delta}}{1 - \left(\frac{t_F}{t_T}\right)^2} \right]^{1/2}$$

$$\Rightarrow \frac{a}{R_s} = \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}}$$

2.0

Expressions lacking the use of approximation $R_s/a \ll 1$, but otherwise correct will get a penalty of 1.0.

If penalty has already been imposed in part (T12.6), no further penalty for lack of approximation.

(T12.8) Combine the results of the wobble method and the transit method to determine the stellar mean density $\rho_s \equiv \frac{M_s}{4\pi R_s^3/3}$ in terms of t_T , t_F , Δ and P . 6

Solution:

From part (T12.7)

$$a = R_s \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}}$$

From part (T12.1)

$$a = \left(\frac{GM_s}{4\pi^2} P^2 \right)^{1/3}$$

Combining,

$$\frac{GM_s}{4\pi^2} P^2 = \left[R_s \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}} \right]^3$$

3.0

Identifying the two equations for combining gets credit. No credit for writing only one equation or irrelevant equations.

$$\Rightarrow \rho_s \equiv \frac{M_s}{4\pi R_s^3/3} = \frac{3}{4\pi} \frac{4\pi^2}{P^2 G} \frac{8P^3 \Delta^{3/4}}{\pi^3 (t_T^2 - t_F^2)^{3/2}}$$

2.0

$$\Rightarrow \rho_s = \frac{24}{\pi^2 G} \frac{P(\Delta)^{3/4}}{(t_T^2 - t_F^2)^{3/2}}$$

1.0

Rocky or gaseous:

Let us consider an edge-on ($i = 90^\circ$) star-planet system (circular orbit for the planet), as seen from the Earth. It is known that the host star is of mass $1.00M_\odot$. Transits are observed with a period (P) of 50.0 days and total transit duration (t_T) of 1.00 hour. The transit depth (Δ) is 0.0064. The same system is also observed in the wobble method to have a maximum line of sight velocity of 0.400 m s^{-1} .

(T12.9) Find the orbital radius a of the planet in units of AU and in metres.

2

Solution:

From Kepler's third law (with same mass of host star):

$$\frac{a}{a_\oplus} = \left(\frac{P}{P_\oplus} \right)^{2/3}$$

1.0

$$a = \left(\frac{50.0}{365.242} \right)^{2/3} \times 1 \text{ AU} = \boxed{0.266 \text{ AU}}$$

0.5

$$= 0.266 \times 1.496 \times 10^{11} \text{ m} = \boxed{3.97 \times 10^{10} \text{ m}}$$

0.5

(T12.10) Find the ratio t_F/t_T of the system.

2

Solution:

Edge-on $\Rightarrow b = 0$

1.0

$$\frac{t_F}{t_T} = \left[\frac{(1 - \sqrt{\Delta})^2 - b^2}{(1 + \sqrt{\Delta})^2 - b^2} \right]^{1/2} = \frac{1 - \sqrt{\Delta}}{1 + \sqrt{\Delta}} = \boxed{0.8519}$$

1.0

- (T12.11) Obtain the mass M_p and radius R_p of the planet in terms of the mass ($M_\oplus$) and radius ($R_\oplus$) of the Earth respectively. Is the composition of the planet likely to be rocky or gaseous? Tick the box for ROCKY or GASEOUS in the Summary Answersheet. 8

Solution:

From parts (T12.1) and (T12.9)

$$v_p = \sqrt{\frac{GM_\odot}{a}} = \sqrt{\frac{6.674 \times 10^{-11} \times 1.989 \times 10^{30}}{3.97 \times 10^{10}}} = 57.798 \text{ km s}^{-1}$$

1.0

Assumption of small planet ($M_p \ll M_s$) is valid because Δ is very small; less dense planet would make the assumption stronger!

$$M_p = \frac{M_s v_0}{v_p} = \frac{1.989 \times 10^{30} \times 0.400}{5.7798 \times 10^4 \times 5.972 \times 10^{24}} M_\oplus = \boxed{2.30 M_\oplus}$$

2.0

From part (T12.4),

$$R_p = R_s \sqrt{\Delta}$$

From part (T12.7),

$$\frac{a}{R_s} = \frac{2P\Delta^{1/4}}{\pi(t_T^2 - t_F^2)^{1/2}} = \frac{2P\Delta^{1/4}}{\pi t_T(1 - (t_F/t_T)^2)^{1/2}}$$

$$\therefore R_s = \frac{a\pi t_T(1 - (t_F/t_T)^2)^{1/2}}{2P\Delta^{1/4}}$$

Combining,

$$\begin{aligned} R_p &= \frac{a\pi t_T(1 - (t_F/t_T)^2)^{1/2}\Delta^{1/2}}{2P\Delta^{1/4}} \\ &= \frac{a\pi t_T(1 - (t_F/t_T)^2)^{1/2}\Delta^{1/4}}{2P} \\ &= \frac{3.97 \times 10^{10} \times \pi \times \frac{1}{24} \times (1 - 0.8519^2)^{1/2} \times (0.0064)^{1/4}}{2 \times 50.0 \times 6.371 \times 10^6} R_\oplus \\ &= \boxed{1.21 R_\oplus} \end{aligned}$$

2.0

1.0

Mean density

$$\rho_p = \frac{M_p}{4\pi R_p^3/3} = \frac{2.30}{(1.21)^3} \rho_\oplus = 1.3\rho_\oplus$$

1.0

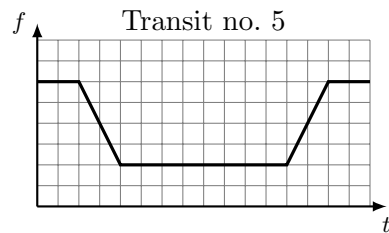
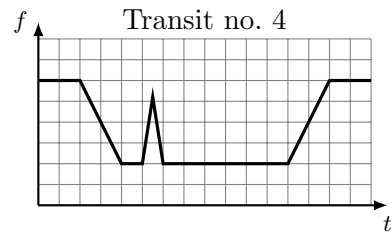
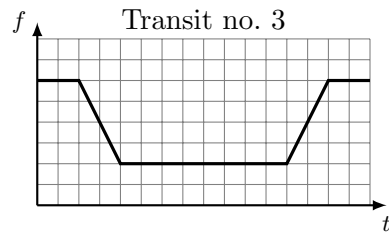
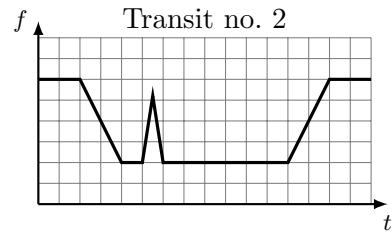
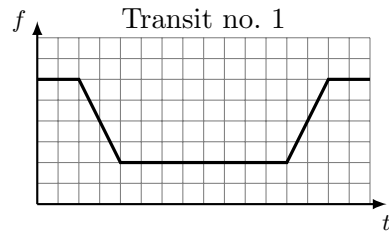
Since mean density is higher than that of Earth, the planet is Rocky.

1.0

Transit light curves with starspots and limb darkening:

- (T12.12) Consider a planetary transit with $i = 90^\circ$ around a star which has a starspot on its equator, comparable to the size of the planet, R_p . The rotation period of the star is $2P$. Draw schematic diagrams of the transit light curve for five successive transits of the planet (in the templates provided in the Summary Answersheet). The no-transit flux for each transit may be normalised to unity independently. Assume that the planet does not encounter the starspot on the first transit but does in the second. 4

Solution:

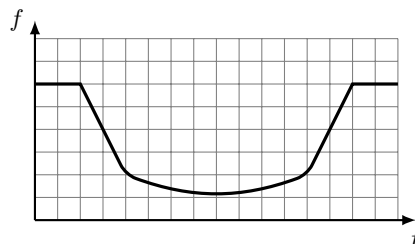


- No change in the first: 0.5
- Spike in second (width and phase of spike is arbitrary): 1.0
- Height of spike (almost) equal to maximum dip: 0.5
- No change in third: 0.5
- Spike again in fourth: 0.5
- Same phase of spike in second and fourth: 0.5
- No change in 5th: 0.5

(T12.13) Throughout the problem we have considered a uniformly bright stellar disc. However, real stellar discs have limb darkening. Draw a schematic transit light curve when limb darkening is present in the host star.

2

Solution:



Non-flat bottom with central minimum gets 2.0. Curvature of ingress and egress are tolerated.

SHORT PROBLEMS: MARKING SCHEME

1. Several exoplanets have been observed in the Gliese 876 system ($M_G = 0.33 \pm 0.03 M_\odot$) as given in the following table,

Gliese System	Mass	Semi Major Axis (AU)
Gliese 876 b	$2.276 M_J$	0.2083
Gliese 876 c	$0.714 M_J$	0.1296
Gliese 876 d	$6.8 M_\oplus$	0.0208
Gliese 876 e	$15 M_\oplus$	0.334

where $M_\odot$ is mass of Sun, M_J is mass of Jupiter ($M_J = 1.89813 \times 10^{27} \text{ kg}$), and $M_\oplus$ is mass of Earth. Assume that all these planets revolve around Gliese 876 in the same direction. Two planets are said to be in resonant orbits if the synodic period of one planet with respect to the other planet is an integer multiple of the orbital period of the second planet.

Find if any of the exoplanets of Gliese 876 system may have resonant orbits.

Answer and Marking Scheme:

1

Determine planet periods

Synodic period of planet 1 and 2 (assume $P_1 < P_2$) is $\frac{1}{P_s} = \frac{1}{P_1} - \frac{1}{P_2} \rightarrow P_s = \frac{P_1 P_2}{P_2 - P_1}$

Resonant orbit happens if

$$P_s = m P_1 \rightarrow \frac{P_1 P_2}{P_2 - P_1} = m P_1 \rightarrow m = \frac{P_2}{P_2 - P_1} > 1, \text{ integer}$$

Using the data of the Gliese system we have

Gliese System	Mass (kg)	Semi Major Axis (m)	Period (Earth, days) $P = 2\pi \sqrt{\frac{a^3}{GM_G}}$
Gliese 876 star	6.64359×10^{29}		
Gliese 876 b	4.31938×10^{27}	3.1161×10^{10}	60.07568
Gliese 876 c	1.35564×10^{27}	1.9388×10^{10}	29.48304
Gliese 876 d	4.079×10^{25}	3.1116×10^9	1.8956
Gliese 876 e	8.7194×10^{25}	5.0011×10^{10}	122.1432

50

2	Calculate resonant orbit for each planet From the results above we see that - Gliese 876 c and Gliese 876 b have resonant orbit $m = \frac{P_b}{P_b - P_c} = 1.96 \approx 2$ - Gliese 876 b and Gliese 876 e have resonant orbit $m = \frac{P_e}{P_e - P_b} = 1.9679 \approx 2$	25 25
---	--	---------------------

2. One satellite of a planet has an orbital period of 7 days, 3 hours, 43 minutes, and the semi major axis is 15.3 times the mean radius of the planet. The Moon has an orbital period of 27 days, 7 hours, 43 minutes and the semi major axis is 60.3 times the Earth's mean radius. Assume that the mass of the moon and the satellite is negligible compared to the mass of the planet. Calculate the ratio of the planet's mean density to that of the Earth.

Answer and Marking Scheme:

First Version

1	Kepler third Law for satellite and Moon $\frac{P_S^2}{a_S^3} = \frac{P_S^2}{(15.3 R_P)^3} = \frac{4\pi^2}{GM_P};$ $\frac{P_M^2}{a_M^3} = \frac{P_M^2}{(60.3 R_E)^3} = \frac{4\pi^2}{GM_E};$	30
2	Express mass in term of mass density: $\frac{P_S^2}{(15.3 R_P)^3} = \frac{4\pi^2}{GM_P} = \frac{4\pi^2}{G\rho_P \frac{4}{3}\pi R_P^3}$ $\frac{P_M^2}{(60.3 R_E)^3} = \frac{4\pi^2}{GM_E} = \frac{4\pi^2}{G\rho_E \frac{4}{3}\pi R_E^3}$	30
3	Evaluate ratio of mass-density: $\frac{\rho_E}{\rho_P} = \frac{60.3^3 \times P_S^2}{15.3^3 \times P_M^2}$ $\frac{\rho_E}{\rho_P} = 0.238 \approx 0.24$	30 10

Second Alternate Version

- The period of the Satellite: $P_S = 7 \text{ days}, 3 \text{ hours}, 43 \text{ minutes} = 0.0196 \text{ years}$,
- Mean orbital radius of the Satellite : $a_S = 15.3 R_P$; R_P : radius of the Planet.
- The period of the Moon: $P_M = 27 \text{ days}, 7 \text{ hours}, 43 \text{ minutes} = 0.0749 \text{ years}$,
- Mean orbital radius of the Moon : $a_M = 60.3 R_E$; R_E : radius of the Earth.
- Mass of the Sun (M)
- Kepler's Third Law : $G(M + m) = 4\pi^2 \frac{a^3}{T^2} \rightarrow$
 - Mass of the Planet: m_P
 - Radius of the Planet : R_P

- Semi major axis of the Planet = Mean orbital radius of the Planet: a_P
- Semi major axis of the Satellite = Mean orbital radius of the Satellite: a_S
- Semi major axis of the Moon = Mean orbital radius of the Moon: a_M
- Mass of the Earth : m_E
- Radius of the Earth: R_E
- Semi major axis of the Earth = Mean orbital radius of the Earth: a_E

$$\text{Planet – Satellite: } G(m_P + m_S) = 4\pi^2 \frac{a_S^3}{P_S^2}, m_P \gg m_S \rightarrow G(m_P) = 4\pi^2 \frac{a_S^3}{P_S^2} \quad (1) \quad (5)$$

$$\text{The Earth–the Moon: } G(m_E + m_M) = 4\pi^2 \frac{a_M^3}{P_M^2}, m_E \gg m_M \rightarrow G(m_E) = 4\pi^2 \frac{a_M^3}{P_M^2} \quad (2) \quad (5)$$

$$\text{Planet–Sun : } G(M + m_P) = 4\pi^2 \frac{a_P^3}{P_P^2}, M \gg m_P \rightarrow G(M) = 4\pi^2 \frac{a_P^3}{P_P^2} \quad (3) \quad (5)$$

$$\text{Earth–Sun : } G(M + m_E) = 4\pi^2 \frac{a_E^3}{P_E^2}, M \gg m_E \rightarrow G(M) = 4\pi^2 \frac{a_E^3}{P_E^2} \quad (4) \quad (5)$$

From equation (1) and (3):

$$\frac{m_P}{M} = \frac{4\pi^2 \frac{a_S^3}{P_S^2}}{4\pi^2 \frac{a_P^3}{P_P^2}} = \frac{(a_S)^3 (P_P)^2}{(a_P)^3 (P_S)^2} = \frac{(15.3R_P)^3 (P_P)^2}{(a_P)^3 (P_S)^2} = \frac{(15.3R_P)^3}{(a_P)^3} \frac{(P_P)^2}{(0.0196)^2} \quad (20)$$

From equation (2) and (4):

$$\frac{m_E}{M} = \frac{4\pi^2 \frac{a_M^3}{P_M^2}}{4\pi^2 \frac{a_E^3}{P_E^2}} = \frac{(a_M)^3 (P_E)^2}{(a_E)^3 (P_M)^2} = \frac{(60.3R_E)^3 (P_E)^2}{(a_E)^3 (P_E)^2} = \frac{(60.3R_E)^3}{(a_E)^3} \frac{(1)^2}{(0.0749)^2} = \frac{(60.3R_E)^3}{(1)^3} \frac{(1)^2}{(0.0749)^2} \quad (20)$$

→

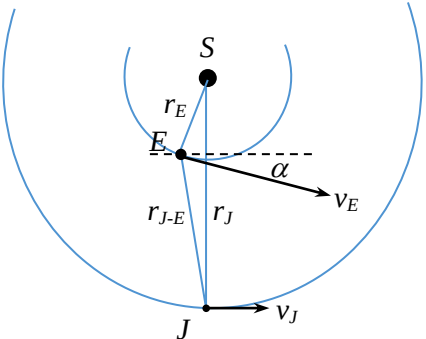
$$\frac{m_E}{m_P} = \frac{\frac{(15.3R_P)^3 (P_P)^2}{(a_P)^3 (0.0196)^2}}{\frac{(60.3R_E)^3 (1)^2}{(1)^3 (0.0749)^2}} = \frac{(15.3R_P)^3 / (0.0196)^2}{(60.3R_E)^3 / (0.0749)^2} \quad (20)$$

→

$$\frac{\rho_P}{\rho_E} = \frac{m_P / \text{Vol}_P}{m_E / \text{Vol}_E} = \frac{m_P / \frac{4}{3}\pi(R_P)^3}{m_E / \frac{4}{3}\pi(R_E)^3} = \frac{(15.3R_P)^3 / (0.0196)^2}{(60.3R_E)^3 / (0.0749)^2} \cdot \frac{\frac{4}{3}\pi(R_E)^3}{\frac{4}{3}\pi(R_P)^3} = \frac{(15.3)^3}{(60.3)^3} \frac{(0.0196)^2}{(0.0749)^2} = 0.24 \quad (20)$$

3. On 27 May 2015 at 02:18:49, the occultation of the star HIP 89931 (δ –Sgr) by the asteroid 1285 Julietta was observed from Borobudur temple, which was located at the center of the asteroid shadow path. It lasted for only 6.201 s. Assume that Earth's orbit is circular and the orbit of Julietta is on the ecliptic plane and revolves in the same direction as Earth. At the occultation, Julietta is near its aphelion. At the time of the occultation the distances of Julietta from the Sun and the Earth are 3.076 AU and 2.156 AU respectively. Find the approximate diameter of asteroid Julietta, if the semi major axis of Julietta is $a = 2.9914$ AU.

Answer and Marking Scheme:

1	Calculate the Earth revolution speed Earth revolution speed is $v_E = \frac{2\pi r_E}{T_E} = \frac{2\pi a_E}{T_E} = 29805 \text{ m/s}$	20
2	Calculate Julietta revolution speed Julietta revolution is (since the given data is semimajor axis, the orbit of Julietta is ellipse) $v_J = \sqrt{GM \left(\frac{2}{r_J} - \frac{1}{a} \right)} = 16742,9 \text{ m/s}$	25
3	Calculate the approximate dimension of Julietta  From the geometry $\cos \alpha = \frac{r_E^2 + r_J^2 - r_{J-E}^2}{2r_E r_J} = 0.995$ Thus, Julietta's relative projected speed to Earth is $v_{J,rel} = v_E - v_J \cos \alpha = 13984.16 \text{ m/s}$ Hence, the approximate dimension of Julietta is $l_J = v_{J,rel} t_{occil} = 86701.8 \text{ m}$	40 15

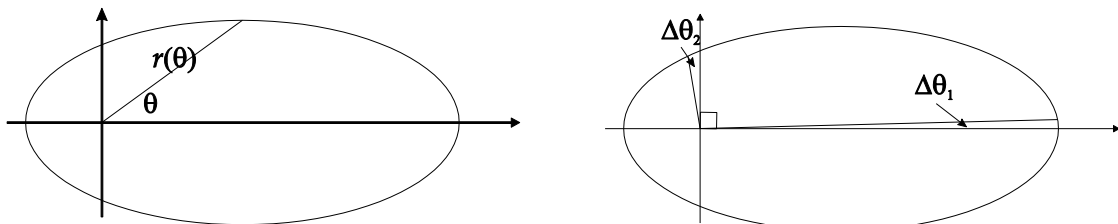
4. Let us assume that an observer using a hypothetical, far-infrared, Earth-sized telescope (wavelength range 20 to 640 μm) found a static and neutral supermassive black hole with a mass of $2.1 \times 10^{10} M_{\odot}$. Determine the maximum distance at which this black hole can be resolved by the observer.

Answer and Marking Scheme:

1	Identify the relationship between the telescope's resolution and the angular size of black hole This problem can be solved by using relationship between the telescope's resolution and the angular size of black holes which can be seen in this equation: $\phi_{\text{telescope}} \leq \phi_{\text{blackhole}} \Rightarrow 1.2 \frac{\lambda}{D_{\text{telescope}}} \leq \frac{\text{Diameter of event horizon}}{\text{Observer Distance to galactic's the center}}$ $1.2 \frac{\lambda}{D_{\text{telescope}}} \leq \frac{2R_{\text{blackhole}}}{d}$	25
2	Identify the relation for determining the distance Then, it is assumed that the radius of black hole is the Schwarzschild radius; and the telescope is also Earth-sized telescope. Therefore: $1.2 \frac{\lambda}{2R_{\text{Earth}}} \leq \frac{2R_{\text{Schwarzschild,BH}}}{d}$ $d \leq \frac{4 \times R_{\text{Schwarzschild,BH}} \times R_{\text{Earth}}}{1.2 \times \lambda} \Rightarrow d \leq \frac{4 \times M_{\text{Black Hole}} \times R_{\text{Schwarzschild,Sun}} \times R_{\text{Earth}}}{1.2 \times M_{\text{Sun}} \times \lambda}$ $d \leq \frac{4 \times 2.1 \times 10^{10} M_{\text{Sun}} \times R_{\text{Schwarzschild,Sun}} \times R_{\text{Earth}}}{1.2 \times M_{\text{Sun}} \times \lambda} \Rightarrow d \leq \frac{8.4 \times 10^{10} \times R_{\text{Schwarzschild,Sun}} \times R_{\text{Earth}}}{1.2 \times \lambda}$	50
3	Calculating the maximum-observer distance To find the maximum distance, then we should use the minimum λ. Therefore: $d_{\text{max}} = \frac{8.4 \times 10^{10} \times R_{\text{Schwarzschild,Sun}} \times R_{\text{Earth}}}{1.2 \times \lambda_{\text{min}}} = \frac{8.4 \times 10^{10} \times 2.95 \times 10^3 \times 6.3708 \times 10^6}{1.2 \times 20 \times 10^{-6}} m = 6.6 \times 10^{25} m$	25

5. An observer is trying to determine an approximate value of the orbital eccentricity of a man-made satellite. When the satellite was at apogee, it was observed to have moved by $\Delta\theta_1 = 2'44''$ in a short time. When the radius vector connecting Earth and the satellite is perpendicular to the major axis (true anomaly is equal to 90°), within the same duration of time, it was observed to have moved by $\Delta\theta_2 = 21'17''$. Assume that the observer is located at the center of the Earth. Find an approximate value of the eccentricity of the satellite's orbit.

Answer and Marking Scheme:

1	Identify expression for $r(\theta)$. We know that $r(\theta) = \frac{a(1-e^2)}{1-e\cos\theta}$. Hence $r(0) = \frac{a(1-e^2)}{1-e} = a(1+e)$ $r\left(\frac{\pi}{2}\right) = \frac{a(1-e^2)}{1-e \times 0} = a(1-e^2)$ 	30
2	Estimate area of sectors using area of triangles. Estimate the area of swept out sectors as the area of sectors of circles $A(S_1) \approx 0.5 \times \Delta\theta_1 \times (r(0))^2 = \Delta\theta_1(a(1+e))^2$ $A(S_2) \approx 0.5 \times \Delta\theta_2 \times \left(r\left(\frac{\pi}{2}\right)\right)^2 = \Delta\theta_2(a(1-e^2))^2$	20
3	Use Kepler's second law to find a relation between the ratio $\frac{\Delta\theta_1}{\Delta\theta_2}$ and the eccentricity e. Use Kepler's second law to obtain that $A(S_1) = A(S_2)$ $\Delta\theta_1 \times (a(1+e))^2 = \Delta\theta_2 \times (a(1-e^2))^2$ Thus, $\frac{\Delta\theta_1}{\Delta\theta_2} = \left(\frac{1-e^2}{1+e}\right)^2 = (1-e)^2 = \frac{2'44''}{21'17''} \approx 0.12843$	30
4	Obtain an estimate value of the eccentricity. Thus, the eccentricity is $e \approx 1 - \sqrt{0.12843} = 0.64$	20

6. At the start of every observation, a radio telescope is pointed at a point-source calibrator that has a known flux density of 21.86 Jy outside the Earth's atmosphere. However, on a certain date, the measured flux density of the calibrator source was 14.27 Jy. If the calibrator source was at an altitude of 35 degrees, estimate the zenith atmospheric optical depth, τ_z .

Answer and Marking Scheme:

1	Atmospheric optical depth τ_A can be determined from the measured flux, that is: $S_{\text{meas}} = \exp(-\tau_A) S_{\text{real}}$ $14.27 = \exp(-\tau_A) \times 21.86 \rightarrow \exp(-\tau_A) = 14.27/21.86 = 0.65$ Then $\tau_A = -\ln(0.65) = 0.43$	50
2	Now we have: $\tau_A = \tau_z \sec z$ $\tau_z = \tau_A \cos z$ $= 0.43 \times \cos (90 - 35)^\circ = 0.25$ where z is zenith angle.	50

7. A galaxy at the boundary of a galaxy cluster of radius 10 Mpc is expected to escape from the cluster if it has an initial velocity of at least 700 km/s relative to the center of the cluster. Calculate the density of the cluster.

Answer and Marking Scheme:

1	We have the total energy of the escaping galaxy: $E_T = E_K + E_P = \frac{mV^2}{2} - \frac{GMm}{R}$	15
2	Where V is velocity of galaxy and R radius of galaxy cluster Galaxy will escape from galaxy cluster if the total energy, $E_T = 0$	10
3	$E_T = 0 \rightarrow \frac{mV^2}{2} - \frac{GMm}{R} = 0$ $V^2 = \frac{2GM}{R}$	15
4	Escape velocity $V^2 = \frac{2G\left(\frac{4\pi}{3}R^3\rho\right)}{R}$	20
5	Density of cluster $\rho = \frac{3V^2}{8\pi GR^2}$	15
6	$V=700$ km/s $R=10$ Mpc $G=6.6378 \times 10^{-11} \text{Nm}^2\text{kg}^{-2}$ $\rho=9.20121 \times 10^{-27} \text{kg/m}^3$	25

8. A strong continuum radio signal from a celestial body has been observed as a burst with a very short duration of 700 μs . The observed flux density at a frequency of 1660 MHz is measured to be 0.35 kJy. If the distance from the source is known to be 2.3 kpc, estimate the brightness temperature of this source.

Answer and Marking Scheme:

1	During 700 μs, the radio wave travels about $r = c t = 3 \times 10^{10} \times 700 \times 10^{-6} = 2.1 \times 10^7$ cm. The region from where the burst originates must be no larger than the distance that light can travel during the duration of the burst. So we estimate that r is the size of the source. ($r = 2.1 \times 10^7$ cm, $R = 2.3$ kpc) Flux density observed at 1660 MHz is $S_{1660\text{MHz}} = 0.35 \text{ kJy}$ $= 0.35 \times 10^3 \times 10^{-23} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$ $= 3.5 \times 10^{-21} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$	10 10
2	The solid angle subtended by this source of radiation is $\Omega = \pi (r/R)^2 = 3.14 (2.1 \times 10^7 / 2.3 \times 10^3 \times 3.086 \times 10^{18})^2$ $= 2.75 \times 10^{-29} \text{ sr}$	30
3	The flux density is related to the total brightness by the relation: $S_{1660\text{MHz}} = B_{1660\text{MHz}} \Omega$ while at this frequency, the total brightness can be approximated from the Rayleigh-Jeans formula: $B_{1660\text{MHz}} = 2kT_b \nu^2 / c^2$ where T_b is the brightness temperature. Then we have: $T_b = S_{1660\text{MHz}} c^2 / 2k \nu^2 \Omega$ $T_b = [3.5 \times 10^{-21} \times (3 \times 10^{10})^2] / [2 \times (1.38 \times 10^{-16}) \times (1660 \times 10^6)^2 \times (2.75 \times 10^{-29})]$ $= 1.5 \times 10^{26} \text{ K}$	20 20 10

9. Assume that the Sun is a perfect blackbody. Venus is also assumed to be a blackbody, with temperature T_V , and it is in thermal equilibrium (i.e. it is radiating about as much energy as it receives from the Sun) at its orbital distance of 0.72 AU. Suppose that at closest approach to Earth, Venus has an angular diameter of about 66 arcsec. What is the flux density of Venus at the closest approach to Earth as observed by a radio telescope at an observing frequency of 5 GHz?

Answer and Marking Scheme:

1	Solar luminosity is $L_{\odot} = 4\pi R_{\odot}^2 \sigma T_{\odot}^4$ $= 4 \times 3.14 \times (6.96 \times 10^{10})^2 (5.67 \times 10^{-5}) (5780)^4 \text{ ergs/sec}$ $= 3.9 \times 10^{33} \text{ ergs/sec}$	10
2	Flux of the Sun intercepted by Venus is $L_V = (A_{V(\text{projected})} / A_{(\text{sphere}, 0.72\text{au})}) \times L_{\odot}$ Also, $L_V = 4\pi R_V^2 \sigma T_V^4$ So, $4\pi R_V^2 \sigma T_V^4 = (\pi R_V^2 / 4\pi d^2) \times L_{\odot} = (\pi R_V^2 / 4\pi d^2) \times 4\pi R_{\odot}^2 \sigma T_{\odot}^4$	30
3	Therefore, $T_V^4 = (R_{\odot}^2 / 4d^2) \times T_{\odot}^4 = [(6.957 \times 10^5 \text{ km})^2 / 4 \times (0.72 \times 149.6 \times 10^6)^2] \times (5780)^4$ $= (483998490000 / 46407499776000000) \times (5780)^4 = (4.84 \times 10^{11} / 4.64 \times 10^{16}) \times (5780)^4$ $= 1.0429316216908190111241385226915 \times 10^{-5} \times 1114577187760656$ $= 1.04 \times 10^{-5} \times 1.11 \times 10^{15}$ $= 11624277939.308134327737156481455 = 1.1624 \times 10^{10}$ $T_V = 328.5 \text{ K}$	20
4	The flux density of Venus at closest approach to Earth at an observing frequency of 5 GHz: ($1'' = 4.84 \times 10^{-6} \text{ rad}$) $S_{5\text{GHz}} = B_{5\text{GHz}} \Omega = (2kT_b v^2 / c^2) \Omega \quad (\rightarrow T_b = T_V)$ $= (2 \times 1.38 \times 10^{-16} \times 328.5 \times (5 \times 10^9)^2 \times \pi (66 \times 4.84 \times 10^{-6})^2) / (3 \times 10^{10})^2$ $= 8.07 \times 10^{-22} \text{ ergs s}^{-1} \text{ cm}^{-2} \text{ Hz}^{-1}$ $= 80.7 \text{ Jy}$	40

10. A molecular hydrogen cloud is known to have a temperature $T = 115$ K. The hydrogen atoms (assumed spherical) have (covalent) radius $r_H = 0.37 \times 10^{-10} m$ and the separation centre-to-centre distance between the two atoms is $d_{H_2} = 0.74 \times 10^{-10} m$. Assume that the molecules are in thermal equilibrium. Estimate the frequency at which they will radiate due to molecular rotational excitation.

Answer and Marking Scheme:

1	The molecules rotate with angular velocity ω and for spherical molecules, their moment of inertia is $I_H = I_{cm} + m_H r_H^2$ $= \frac{2}{5} m_H^2 + m_H r_H^2 = \frac{7}{5} m_H r_H^2$ For H_2 molecule, yield $I_{H_2} = 2I_H = \frac{14}{5} m_H r_H^2,$ Thus $I = 6.401 \times 10^{-48} \text{ Kg m}^2$	30
2	In thermal equilibrium(only 2 rotational degrees of freedom), we have: $\frac{1}{2} I \omega^2 = kT$	30
3	So we can estimate: $\omega = \sqrt{2kT/I} = (2 \times 1.38 \times 10^{-16} \times 115 / 6.40 \times 10^{-48})^{1/2} = 2.23 \times 10^{13} s^{-1}$ $\nu = \frac{\omega}{2\pi} = 3.55 \times 10^{13} \text{ Hz}$	40

11. The mass density of an object is inversely proportional to the radial distance from the center of the object with a factor of proportionality $\alpha = 5.0 \times 10^{13} \text{ kg/m}^2$. If the escape velocity at the surface of the object is $v_0 = 1.5 \times 10^4 \text{ m/s}$, calculate the total mass of the object.

	Answer	
1	$\frac{1}{2}mv_0^2 - \frac{GmM}{R} = 0 \rightarrow R = \frac{2GM}{v_0^2}$	20
2	$\Delta m(r) = \rho 4\pi r^2 \Delta r = 4\pi\alpha r \Delta r \rightarrow M = 2\pi\alpha R^2$ $(dm(r) = \rho 4\pi r^2 dr = 4\pi\alpha r dr \rightarrow M = 2\pi\alpha R^2)$	30
3	$M = \alpha 2\pi R^2 = \alpha 2\pi \frac{4G^2 M^2}{v_0^4} \rightarrow M = \frac{v_0^4}{8\pi\alpha G^2}$	30
4	$M = \frac{v_0^4}{8\pi\alpha G^2} = \frac{(1.5 \times 10^4 \text{ ms}^{-1})^4}{8\pi(5 \times 10^{13} \text{ kgm}^{-2})(6.67 \times 10^{-11} \text{ kg}^{-1}\text{m}^3\text{s}^{-2})^2} = 9.1 \times 10^{21} \text{ kg}$	20

12. A proton with a kinetic energy of 1GeV propagates out from the surface of the Sun towards the Earth. Neglecting the magnetic field of the Sun, calculate the travel time of the proton as seen from the Earth.

	Answer	
1	$E_0 = m_0 c^2 = 941\text{ MeV}$	20
2	$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{K}{E_0} + 1 = \frac{1000\text{ MeV}}{941\text{ MeV}} + 1 = 2.062$	20
3	$1 - \frac{v^2}{c^2} \approx \frac{1}{4} \rightarrow v \approx 0.87c$	30
4	$\begin{aligned} \text{Elapse time} &= \frac{\text{Earth} - \text{Sun distance}}{\text{speed}} = \frac{1.50 \times 10^{11}\text{ m}}{0.87 \times (3 \times 10^8)\text{ m/s}} \\ &= 9.6\text{ minutes} \end{aligned}$	30

13. Volcanic activity on Io, whose rotation period synchronizes with its orbital period, was proposed to be the result of tidal heating mainly from Jupiter. The resultant tidal force on a body is the difference in gravitational force experienced by the near and far sides of that body due to another body. Measurements of the surface distortion of Io via satellite radar altimeter mapping indicate that the surface rises and falls by up to 100m during one-half orbit. Only the surface layers will move by this amount. Interior layers within Io will move by a smaller amount, and thus we assume that on average the entire mass of Io is moved through 50m. Assume that Io is considered as two hemispheres each treated as a point mass. Calculate the *average power* of the tidal heating on Io.

Hint: you can use the following approximation $(1+x)^n \approx 1+nx$ for small x .

The mass of Io is $m_{Io} = 8.931938 \times 10^{22} \text{ kg}$

The perijove distance is $r_{peri} = 420000 \text{ km}$

The apojove distance is $r_{apo} = 423400 \text{ km}$

The orbital period of Io is 152853 s

The radius of Io is $R_{Io} = 1821.6 \text{ km}$.

Answer and Marking Scheme:

1	Calculate the tidal force The tidal force (assuming the point mass of hemisphere is located at $R_{Io}/2$): $F_{\text{tidal}} = F(r - R_{Io}) - F(r + R_{Io})$ $= -\frac{GM_J \frac{m_{Io}}{2}}{\left(r - \frac{R_{Io}}{2}\right)^2} + \frac{GM_J \frac{m_{Io}}{2}}{\left(r + \frac{R_{Io}}{2}\right)^2} = -GM_J \frac{m_{Io}}{2} \left[\frac{1}{\left(r - \frac{R_{Io}}{2}\right)^2} - \frac{1}{\left(r + \frac{R_{Io}}{2}\right)^2} \right]$ $= -\frac{GM_J \frac{m_{Io}}{2}}{r^2} \left[\frac{\left(\frac{2R_{Io}}{r}\right)}{1 - \left(\frac{R_{Io}}{r}\right)^2} \right]$	40
---	--	----

	The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored, so that $F_{\text{tidal}} = -\frac{GM_J m_{Io} R_{Io}}{r^3}$	10
2	Calculate the difference of the tidal force The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove (furthest from Jupiter) $\Delta F = F_{\text{tidal}}(r_{\text{peri}}) - F_{\text{tidal}}(r_{\text{apo}}) $ $= GM_J m_{Io} R_{Io} \left[\frac{1}{r_{\text{peri}}^3} - \frac{1}{r_{\text{apo}}^3} \right] = 6.65 \times 10^{18} \text{ N}$	15 15
3	Calculate the average power of work done Hence, the average power of the work done on the rock during one-half orbit is $\bar{P} = \frac{\Delta F d}{t_{Io}/2}$ $\bar{P} = 4.35 \times 10^{15} \text{ W}$	10 10

Alternate solution

If assume the mass of the hemisphere is located at the center of mass of a solid hemisphere, $3R_{Io}/8$, then

1	Calculate the tidal force The tidal force (assuming the point mass of hemisphere is located at $3R_{Io}/8$): $F_{\text{tidal}} = F(r - R_{Io}) - F(r + R_{Io})$ $= -\frac{GM_J \frac{m_{Io}}{2}}{\left(r - \frac{3R_{Io}}{8}\right)^2} + \frac{GM_J \frac{m_{Io}}{2}}{\left(r + \frac{3R_{Io}}{8}\right)^2} = -GM_J \frac{m_{Io}}{2} \left[\frac{1}{\left(r - \frac{3R_{Io}}{8}\right)^2} - \frac{1}{\left(r + \frac{3R_{Io}}{8}\right)^2} \right]$ $= -\frac{GM_J \frac{m_{Io}}{2}}{r^2} \left[\frac{\left(\frac{3R_{Io}}{2r}\right)}{1 - \left(\frac{6R_{Io}}{8r}\right)^2} \right]$	40
---	---	----

	The term $\left(\frac{R_{Io}}{r}\right)^2$ is too small and it can be ignored, so that $F_{\text{tidal}} = -\frac{3GM_J m_{Io} R_{Io}}{4r^3}$	10
2	Calculate the difference of the tidal force The difference in the tidal force experienced at perijove (closest to Jupiter) and apojove (furthest from Jupiter) $\Delta F = F_{\text{tidal}}(r_{\text{peri}}) - F_{\text{tidal}}(r_{\text{apo}}) $ $= \frac{3}{4}GM_J m_{Io} R_{Io} \left[\frac{1}{r_{\text{peri}}^3} - \frac{1}{r_{\text{apo}}^3} \right] = 4.97 \times 10^{18} \text{ N}$	15 15
3	Calculate the average power of work done Hence, the average power of the work done on the rock during one-half orbit is $\bar{P} = \frac{\Delta F d}{t_{Io}/2}$ $\bar{P} = 3.25 \times 10^{15} \text{ W}$	10 10

14. Suppose we live in a static and infinitely large universe where the average density of stars is $n = 10^9 \text{ Mpc}^{-3}$ and the average stellar radius is equal to the solar radius. Assume that standard Euclidean geometry holds true in this universe. How far, on average, could you see in any direction before your line of sight strikes a star? Please write your answer in Mpc.

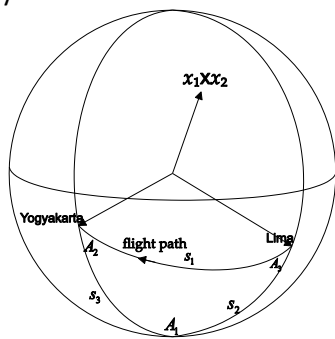
Answer and Marking Scheme:

1.	To decide how far one can see on average in a universe filled with spherical objects of radius R , it is simplest to think of a long cylinder along the line of sight. If an object is closer than R to the line of sight, then the line of sight intersects its surface. For a distance ℓ , the cylindrical volume which would contain such objects is $\pi R^2 \ell$. If the density of objects is n , then the volume that will on average contain one object is defined by $\pi R^2 \ell n = 1$ and the average distance to which we see before our vision is blocked is $\ell = \frac{1}{\pi R^2 n}$	50
2.	The average radius R in Mpc $R = \frac{6.96 \times 10^8 \text{ m}}{3.086 \times 10^{22} \text{ m Mpc}^{-1}} = 2.3 \times 10^{-14} \text{ Mpc}$	25
3.	Then $\ell = \frac{1}{3.14 (2.3 \times 10^{-14} \text{ Mpc})^2 (10^9 \text{ Mpc}^{-3})} = 6.3 \times 10^{17} \text{ Mpc}$	25

15. An airplane was flying from Lima, capital of Peru ($12^{\circ}2'S$ and $77^{\circ}1'W$) to Yogyakarta ($7^{\circ}47'S$ and $110^{\circ}26'E$), near the venue of the 9th IOAA. The airplane chooses the shortest flight path from Lima to Yogyakarta. Find the latitude of the southernmost point of the flight path.

Answer and Marking Scheme:

1st version

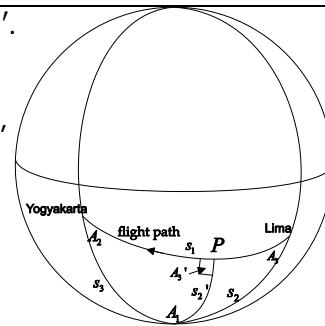
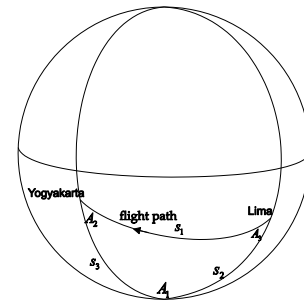
1	Determine coordinates of Lima and Yogyakarta. Assume that the equator is the $z = 0$ plane and the Earth is a unit sphere. The Cartesian coordinates of the Lima is $\begin{aligned} \mathbf{x}_1 &= (x_1, y_1, z_1) \\ &= (\cos(-12^{\circ}2') \cos(282^{\circ}58'48'), \cos(-12^{\circ}2') \sin(282^{\circ}58'48'), \sin(-12^{\circ}2')) \\ &= (0.2197308726, -0.9530236794, -0.2084807188) \end{aligned}$ The Cartesian coordinates of the Yogyakarta is $\begin{aligned} \mathbf{x}_2 &= (x_2, y_2, z_2) = (\cos(-7^{\circ}47') \cos(110^{\circ}26'), \cos(-7^{\circ}47') \sin(110^{\circ}26'), \sin(-7^{\circ}47')) \\ &= (-0.3459009563, 0.9284459898, -0.1354273699) \end{aligned}$	15
2	Determine the normal of the plane defined by Lima, Yogyakarta, and center of the Earth. The normal of the plane containing the two places and the center of the Earth is $\mathbf{x}_1 \times \mathbf{x}_2 = (a, b, c) = (0.3226285776, 0.1018712545, -0.12564355210)$	20
3	Find the equation of the (shortest flight) path from Lima to Yogyakarta. Let the equation of the plane be $ax + by + cz = 0$ or equivalently $y = -a'x - c'z$ where $a' = -3.16702272082062$ and $c' = 1.23335628599724$. For the sake of simplicity, from now let us denote a' and c' as a and c, respectively. Intersection of the Earth and the plane is the large circle $x^2 + (-ax - cz)^2 + z^2 = 1 \text{ or equivalently } (1 + a^2)x^2 + 2acxz + (1 + c^2)z^2 - 1 = 0$ It is also the equation for the shortest flight path from Lima to Yogyakarta. 	25
4	Determine the z-value of the southernmost point of the path (at the point, the discriminant of the equation $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$ equals zero). $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$ The plane $z = m$, it intersects the large circle at at either at one or two points. The number of intersection point at the highest or at the lowest point is one, i.e. when the discriminant is zero. $(1 + a^2)x^2 + 2acmx + (1 + c^2)m^2 - 1 = 0$	30

	The discriminant is $D = (2acm)^2 - 4(1 + a^2)[(1 + c^2)m^2 - 1]$ $= 4 + 4a^2 - (4 + 4a^2 + 4c^2)m^2$ Therefore $D = 0$, if $m = \pm \sqrt{\frac{1+a^2}{1+a^2+c^2}} = 0.937444937497383$. Then choose $z = -0.937444937497383$	
5	Determine the latitude of the southernmost point. Thus, the latitude of the point is $\sin^{-1} z = 1.21521692653748 = 69.6268010651290^\circ = 69^\circ 37' 36'' S$.	10

2nd version (using the Law of Cosines for sides)

Answer and Marking Scheme:

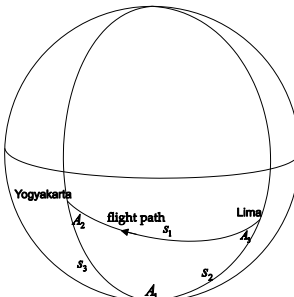
1	Determine length of the sides of the triangle $s_2 = 90^\circ - 12^\circ 2' = 77^\circ 58' = 1.3608 \text{ rad}, s_3 = 90^\circ - 7^\circ 47' = 82^\circ 13' = 1.4350 \text{ rad}; A_1 = (180^\circ - 77^\circ 1') + (180^\circ - 110^\circ 26') = 172^\circ 33' = 3.0116 \text{ rad}$ Using Law of Cosines for sides $\cos s_1 = (\cos s_2)(\cos s_3) + (\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1 = 2.7724 \text{ rad}$	20
2	Determine the angles of the triangle Then use $\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2 = 0.35921 \text{ rad}$. Similarly, use Law of Cosines $\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3 = 0.3642 \text{ rad}$.	30
3	Use Law of Cosines for sides to construct an equation in $\cos s_2'$. If P is the southernmost point, then the angle A_3 is a right angle, $A_3 = \frac{\pi}{2}$, the Law of $\cos s_3 = \cos s_1' \cos s_2' + \sin s_1' \sin s_2' \cos A_3 = \cos s_1' \cos s_2'$ Thus, $\cos s_1' = \frac{\cos s_3}{\cos s_2'} = \frac{0.13538}{\cos s_2'}$ Then substitute it to get	20



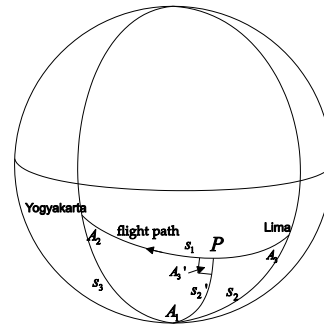
	$\cos s_2' = \cos s_1' \cos s_3 + \sin s_1' \sin s_3 \cos A_2$ $= \frac{\cos s_3}{\cos s_2'} \cos s_3 + \sqrt{1 - \left(\frac{\cos s_3}{\cos s_2'}\right)^2} \sin s_3 \cos A_2$	
4	Solve the equation Let $x = \cos s_2'$. Then $x = \frac{0.018328}{x} + \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ Multiply both sides by x to get $x^2 = 0.018328 + x \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $x^2 - 0.018328 = x \sqrt{\frac{x^2 - 0.018328}{x^2}} \times 0.92756$ $(x^2 - 0.018328)^2 = (x^2 - 0.018328) \times (0.92756)^2$ $x^4 - 0.89702x^2 + 0.016105 = 0$ which is a quadratic equation of x^2 . The roots are ± 0.13538 and ± 0.93739 .	20
5	Determine the Latitude Thus, $s_2' = \cos^{-1} 0.93739 = 0.35574 \text{ rad} = 20^\circ 23'$ or $s_2' = \cos^{-1} 0.13538 = 1.4350 \text{ rad} = 82^\circ 13'$ The latitude is $69^\circ 37'S$ or $7^\circ 47'S$. The second one is impossible because it higher then the latitude of Lima. Thus the answer is $69^\circ 37'S$	10

3rd version (using Napier's rule)

Answer and Marking Scheme:

1	Determine length of the sides of the triangle $s_2 = 90^\circ - 12^\circ 2' = 77^\circ 58' = 1.3608 \text{ rad}, s_3 = 90^\circ - 7^\circ 47' = 82^\circ 13' = 1.4350 \text{ rad}; A_1 = (180^\circ - 77^\circ 1') + (180^\circ - 110^\circ 26') = 172^\circ 33' = 3.0116 \text{ rad}$ Using Law of Cosines for sides $\cos s_1 = (\cos s_2)(\cos s_3) + (\sin s_2)(\sin s_3)(\cos A_1)$ to obtain $s_1 = 2.7724 \text{ rad}$		20
2	Determine the angles of the triangle Then use		30

	$\cos s_2 = (\cos s_1)(\cos s_3) + (\sin s_1)(\sin s_3)(\cos A_2)$ to obtain $A_2 = 0.35921$ rad. Similarly, use Law of Cosines $\cos s_3 = (\cos s_1)(\cos s_2) + (\sin s_1)(\sin s_2)(\cos A_3)$ to get that $A_3 = 0.3642$ rad.	
3	Find the latitude of the point Let P be the southernmost point on the flight path. Since the triangle has a right triangle at P, then use one of Napier's rules $\sin s'_2 = \sin s_3 \sin A_2$ to get $s'_2 = 0.35576 \text{ rad} = 20^\circ 23'$. Therefore, the latitude of P is $90^\circ - 20^\circ 23' = 69^\circ 37' S$	50



IOAA 2013

Volos, Greece



7th International Olympiad
on Astronomy & Astrophysics

VOLOS 2013 HELLAS

THEORY



7th International Olympiad
on Astronomy & Astrophysics
VOLOS 2013 HELLAS



Table of Constants

Mass ($M_{\oplus}$)	5.98×10^{24} kg	Earth
Radius ($R_{\oplus}$)	6.38×10^6 m	
Acceleration of gravity	9.8 m/s^2	
Obliquity of Ecliptic	$23^{\circ}27'$	
Length of Tropical Year	365.2422 mean solar days	
Length of Sidereal Year	365.2564 mean solar days	
Albedo (a)	0.39	
Mass (M_{c})	7.35×10^{22} kg	Moon
Radius (R_{c})	1.74×10^6 m	
Mean distance from Earth	3.84×10^8 m	
Orbital inclination with the Ecliptic	5.14°	
Albedo	0.14	
Apparent magnitude (mean full moon)	-12.74	
Mass ($M_{\odot}$)	1.99×10^{30} kg	Sun
Luminosity ($L_{\odot}$)	3.83×10^{26} W	
Absolute Magnitude ($\mathcal{M}_{\odot}$)	4.72 mag	
Angular diameter	0.5 degrees	
Effective Surface Temperature	5800 K	
Jupiter's orbit semi-major axis	5.204 AU	Jupiter
Jupiter's orbital period	11.8618 yr	
Diameter of human pupil	6 mm	Distances and sizes
1 AU	1.50×10^{11} m	
1 pc	206,265 AU	
Distance from Sun to Barnard's Star	1.83 pc	
Mars orbit semi-major axis	1.52 AU	
Gravitational constant (G)	$6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$	Physical constants
Planck constant (h)	$6.62 \times 10^{-34} \text{ J} \cdot \text{s}$	
Boltzmann constant (k_B)	$1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-2}$	
Stefan-Boltzmann constant (σ)	$5.67 \times 10^{-8} \text{ W} \cdot \text{m}^{-2} \cdot \text{K}^{-4}$	
Hubble constant (H_0)	$72 \text{ km s}^{-1} \text{ Mpc}^{-1}$	
Speed of light (c)	$299,792,458 \text{ m/s}$	
Proton mass	$938.27 \text{ MeV} \cdot \text{c}^{-2}$	
Deuterium mass	$1875.60 \text{ MeV} \cdot \text{c}^{-2}$	
Neutron mass	$939.56 \text{ MeV} \cdot \text{c}^{-2}$	
Helium-3 mass	$2808.30 \text{ MeV} \cdot \text{c}^{-2}$	
Helium-4 mass	$3727.40 \text{ MeV} \cdot \text{c}^{-2}$	

The brightest stars visible from Greece

Star		α (2000)	δ (2000)	m(V)	M(V)	B-V	Spectral
		h m	° '	mag	mag		Type
Alpheratz	α And	00 08	+29 05	2.03	-0.9	-0.10	A0 p
Caph	β Cas	00 09	+59 09	2.26	+1.5	+0.34	F2 IV
Schedar	α Cas	00 40	+21 26	2.22	-1.0	+1.17	K0 II
Diphda	β Cet	00 44	-17 59	2.04	+0.2	+1.04	K0 III
Mirach	β And	01 10	+35 37	2.06	-0.4	+1.62	M0 III
Achernar	α Eri	01 38	-57 15	0.48	-1.6	-0.18	B5 IV
Almach	γ And	02 04	+42 20	2.13	-0.1	+1.20	K2 III
Hamal	α Ari	02 07	+23 28	2.00	+0.2	+1.15	K2 III
Mira	α Cet	02 19	-02 59	2.0	-1.0	+1.42	M6 e
Polaris	α UMi	02 32	+89 16	2.02	-4.6	+0.6	F8 Ib
Algol	β Per	03 08	+40 57	2.2	-0.3	-0.1	B8 V
Mirfak	α Per	03 24	+49 51	1.80	-4.3	+0.48	F5 Ib
Aldebaran	α Tau	04 36	+16 30	0.85	-0.3	+1.54	K5 III
Rigel	β Ori	05 15	-08 12	0.11	-7.0	-0.03	B8 I α
Capella	α Aur	05 17	+46 00	0.08	+0.3	+0.80	G8 III
Bellatrix	γ Ori	05 25	+06 21	1.63	-3.3	-0.22	B2 III
El Nath	β Tau	05 26	+28 36	1.65	-1.6	-0.13	B7 III
Mintaka	δ Ori	05 32	-00 18	2.19	-6.1	-0.21	O9.5II
Alnilam	ϵ Ori	05 36	-01 12	1.70	-6.2	-0.19	B0 I α
Alnitak	ζ Ori	05 41	-01 57	1.79	-5.9	-0.21	O9.5 Ib
Saiph	κ Ori	05 48	-09 40	2.05	-6.8	-0.18	B0.5 I α
Betelgeuse	α Ori	05 55	+07 24	0.50	-5.6	+1.86	M2 Iab
Menkalinan	β Aur	06 00	+44 57	1.90	+0.6	+0.03	A2 IV
Mirzam	β CMa	06 23	-17 57	1.98	-4.5	-0.24	B1 II
Canopus	α Car	06 24	-52 42	-0.73	-4.7	+0.16	F0 Ib
Alhena	γ Gem	06 38	+16 24	1.93	+0.0	+0.00	A0 IV
Sirius	α CMa	06 45	-16 43	-1.45	+1.4	+0.00	A1 V
Adhara	ϵ CMa	06 59	-28 58	1.50	-5.0	-0.22	B2 II
Wezen	δ CMa	07 08	-26 24	1.84	-7.3	+0.67	F8 Ia
Castor	α Gem	07 35	+31 53	1.58	+0.8	+0.04	A1 V
Procyon	α CMi	07 39	+05 14	0.35	+2.7	+0.41	F5 IV
Pollux	β Gem	07 45	+28 01	1.15	+1.0	+1.00	K0 III
Naos	ζ Pup	08 04	-40 00	2.25	-7.0	-0.27	O5.8
—	γ Vel	08 10	-47 20	1.83	-4.0	-0.26	WC 7
Avior	ϵ Car	08 23	-59 30	1.87	-2.1	+1.30	K0 II
—	δ Vel	08 45	-54 43	1.95	+0.1	+0.0	A0 V4
Suhail	λ Vel	09 08	-43 26	2.26	-4.5	+1.69	K5 Ib
Miaplacidu	β Car	09 13	-69 43	1.68	-0.4	+0.00	A0 III

The brightest stars visible from Greece (cont.)

Star		α (2000)		δ	m (V)	M (V)	B-V	Spectral
		h	m	°	mag	mag		Type
Scutulum	ι Car	09	17	-59 16	2.24	-4.5	+0.18	F0 Ib
Alphard	α Hya	09	28	-08 40	1.99	-0.4	+1.43	K3 III
Regulus	α Leo	10	08	+11 58	1.35	-0.6	-0.11	B7 V
Algeiba	γ Leo	10	20	+19 51	2.1	-0.5	+1.12	K0 III
Dubhe	α UMa	11	03	+61 45	1.79	-0.7	+1.06	K0 III
Denebola	β Leo	11	49	+14 34	2.14	+1.6	+0.09	A3 V
Acrux	α Cru	12	27	-63 06	0.9	-3.5	-0.26	B1 IV
Gacrux	γ Cru	12	31	-57 07	1.64	-2.5	+1.60	M3 III
Muhlifain	γ Cen	12	42	-48 58	2.16	-0.5	-0.02	A0 III
Mimosa	β Cru	12	48	-59 41	1.26	-4.7	-0.24	B0 III
Alioth	ε UMa	12	54	+55 57	1.78	-0.2	-0.02	A0 p
Mizar	ζ UMa	13	24	+54 56	2.09	+0.0	+0.03	A2 V
Spica	α Vir	13	25	-11 09	0.96	-3.4	-0.23	B1 V
—	ε Cen	13	40	-53 28	2.30	-3.6	-0.23	B1 V
Alkaid	η UMa	13	48	+49 19	1.86	-1.9	-0.19	B3 V
Hadar	β Cen	14	04	-60 22	0.60	-5.0	-0.23	B1 II
Menkent	θ Cen	14	07	-36 22	2.06	+1.0	+1.02	K0 III
Arcturus	α Boo	14	16	+19 11	-0.06	-0.2	+1.23	K2 IIIp
Rigil Kent	α Cen	14	40	-60 50	-0.1	+4.3	+0.7	G2 V
Kochab	β UMi	14	50	+74 09	2.07	-0.5	+1.46	K4 III
Alphecca	α CrB	15	35	+26 43	2.23	+0.5	-0.02	A0 V
Antares	α Sco	16	29	-26 26	1.0	-4.7	+1.81	M1 Ib
Atria	α TrA	16	49	-69 02	1.93	-0.3	+1.43	K4 III
—	ε Sco	16	50	-34 18	2.29	+0.7	+1.15	K2 III
Shaula	λ Sco	17	34	-37 06	1.62	-3.4	-0.22	B1 IV
Ras-Alhague	α Oph	17	35	+12 34	2.07	+0.8	+0.15	A5 III
—	θ Sco	17	37	-43 00	1.87	-4.5	+0.40	F0 I
Eltanin	γ Dra	17	57	+51 29	2.22	-0.6	+1.52	K5 III
Kaus Australis	ε Sgr	18	24	-34 23	1.83	-1.5	-0.02	B9 IV
Vega	α Lyr	18	37	+38 47	0.04	+0.5	+0.00	A0 V
Nunki	σ Sgr	18	55	-26 18	2.08	-2.5	-0.20	B2 V
Altair	α Aql	19	51	+08 52	0.77	+2.3	+0.22	A7 IV
Sadir	γ Cyg	20	22	+40 15	2.23	-4.7	+0.67	F8 Ib
Peacock	α Pav	20	26	-56 44	1.93	-2.9	-0.20	B3 IV
Deneb	α Cyg	20	41	+45 17	1.25	-7.3	+0.09	A2 Ia
Al Na' ir	α Gru	22	08	-46 58	1.74	+0.2	-0.14	B5 V
—	β Gru	22	42	-46 53	2.20	-1.5	+ 1.6	M3 II
Fomalhaut	α PsA	22	58	-29 37	1.16	+1.9	+0.09	A3 V

The brightest stars of the sky with $m(V) \leq 1.00$

Star		α (2000)		δ (2000)		$m(V)$	$M(V)$	B-V	Spectra type
		h	m	°	'	mag	mag		
Sirius	α CMa	06	45	-16	43	-1.4	+1.4	+0.00	A1 V
Canopus	α Car	06	24	-52	42	-0.7	-4.7	+0.16	F0 Ib
Rigel Kent	α Cen	14	40	-60	50	-0.1	+4.3	+0.7	G2 V
Arcturus	α Boo	14	16	+19	11	-0.0	-0.2	+1.23	K2 IIIp
Vega	α Lyr	18	37	+38	47	0.0	+0.5	+0.00	A0 V
Capella	α Aur	05	17	+46	00	0.08	+0.3	+0.80	G8 III
Rigel	β Ori	05	15	-08	12	0.11	-7.0	-0.03	B8 Ia
Procyon	α CMi	07	39	+05	14	0.35	+2.7	+0.41	F5 IV
Achernar	α Eri	01	38	-57	15	0.48	-1.6	-0.18	B5 IV
Betelgeuse	α Ori	05	55	+07	24	0.50	-5.6	+1.86	M2 Iab
Hadar	β Cen	14	04	-60	22	0.6	-5.0	-0.23	B1 II
Altair	α Aql	19	51	+08	52	0.7	+2.3	+0.22	A7 IV
Aldebaran	α Tau	04	36	+16	30	0.85	-0.3	+1.54	K5 III
Acrux	α Cru	12	27	-63	06	0.9	-3.5	-0.26	B1 IV
Spica	α Vir	13	25	-11	09	0.9	-3.4	-0.23	B1 V
Antares	α Sco	16	29	-26	26	1.0	-4.7	+1.81	M1 Ib



Theoretical Exam - Short Questions

1. What would be the mean temperature on the Earth's surface if we ignore the greenhouse effect, assume that the Earth is a perfect black body and take into account its non-vanishing albedo? Assume that the Earth's orbit around the Sun is circular.

Answer : $T_{\oplus} = T_{\odot} (1 - \alpha)^{1/4} \sqrt{\frac{R_{\odot}}{2r_{\oplus}}} = 246 \text{ K} \quad \text{or} \quad T_{\oplus} = -27^{\circ} \text{ C} \quad (10 \text{ Points})$

(Note: $r_{\oplus}$ = average Earth-Sun distance, the albedo, $a = 0.39$ is given in the *additional material*).

2. Let us assume that we observe a hot Jupiter planet orbiting around a star at an average distance $d = 5 \text{ AU}$. It has been found that the distance of this system from us is $r = 250 \text{ pc}$. What is the minimum diameter, D , that a telescope should have to be able to resolve the two objects (star and planet)? We assume that the observation is done in the optical part of the electromagnetic spectrum ($\lambda \sim 500 \text{ nm}$), outside the Earth's atmosphere and that the telescope optics are perfect.

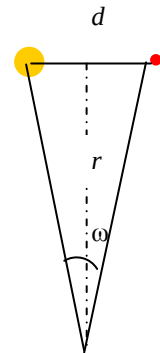
Answer: From the figure we have:

$$\omega(\text{rad}) \approx \frac{d}{r} = \frac{5 \text{ AU}}{250 \text{ pc}} = \frac{5 \times 1.5 \times 10^{11} \text{ m}}{250 \times 3.09 \times 10^{16} \text{ m}} = 9.70 \times 10^{-8} \text{ rad} \quad (5 \text{ Points})$$

Let us assume that D is the minimum diameter of our space telescope.

Its angular resolution is

$$\Delta\theta = 1.22 \times \frac{\lambda}{D} \rightarrow 9.70 \times 10^{-8} \text{ rad} = 1.22 \times \frac{500 \times 10^{-9} \text{ m}}{D} \rightarrow D = 6 \text{ m}$$



(5 Points)

3. It is estimated that the Sun will have spent a total of about $t_1 = 10$ billion years on the main sequence before evolving away from it. Estimate the corresponding amount of time, t_2 , if the Sun were 5 times more massive.

Answer: For the average luminosity of a main sequence star we have: $L \propto M^4$ (where M the initial mass of the star). We assume that the total energy E that the star produces is proportional to its mass $E \propto M$. Therefore the amount of time that the star spends on the main sequence is

$$\text{approximately } t_{\text{MS}} \approx \frac{E}{L} \propto \frac{M}{M^4} \approx M^{-3}.$$

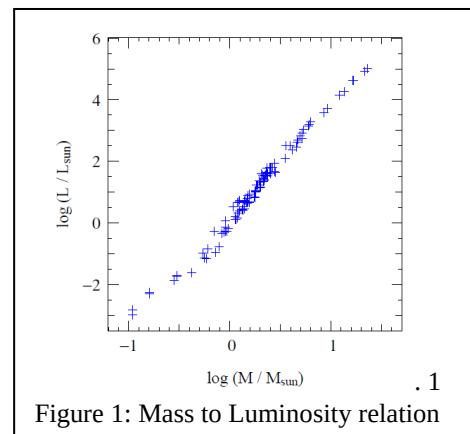


Figure 1: Mass to Luminosity relation

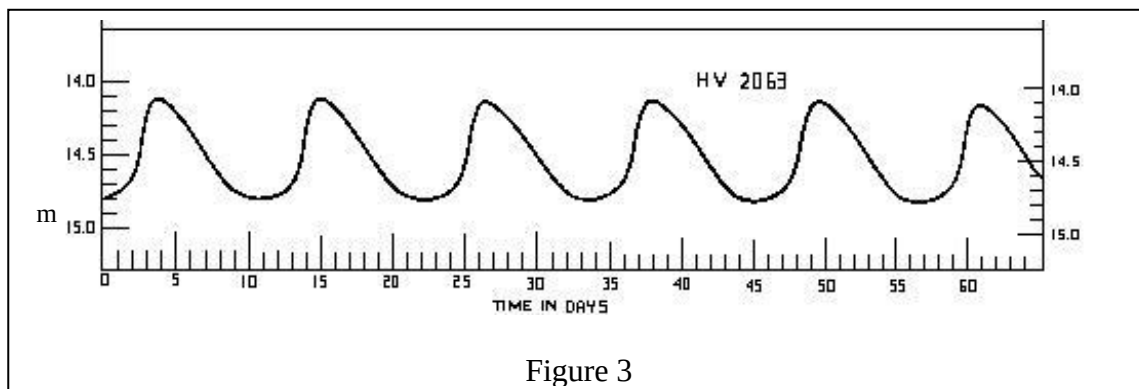
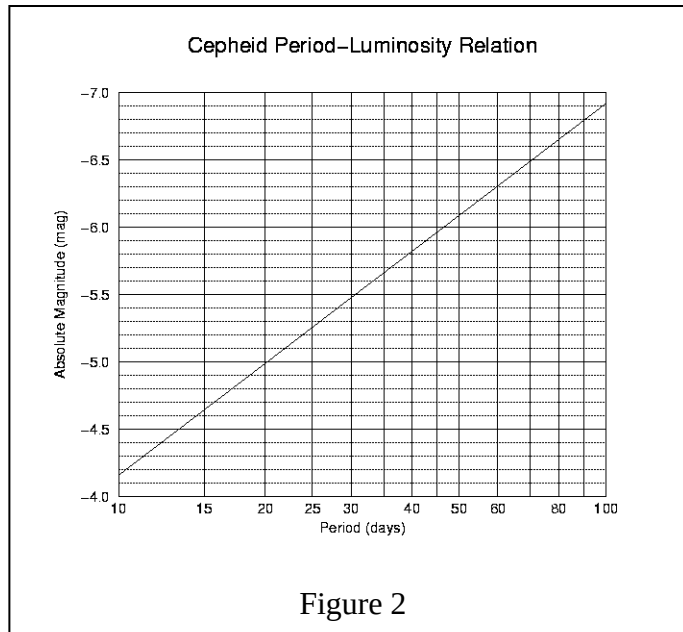
(5 Points)



Therefore, $\frac{t_1}{t_2} \approx \frac{1^{-3}}{5^{-3}} = 5^3 \rightarrow t_2 = \frac{1}{125} \times 10^{10} \text{ yr}$ or $t_2 = 8 \times 10^7 \text{ yr}$.

(5 Points)

4. Figure 2 shows the relation between absolute magnitude and period for classical cepheids. Figure 3 shows the light curve (apparent magnitude versus time in days) of a classical cepheid in a local group galaxy. (a) Using these two figures estimate the distance of the cepheid from us. (b) Revise your estimate assuming that the interstellar extinction towards the cepheid is $A = 0.25 \text{ mag}$.



Answer : (a) From Figure 2, the period of the cepheid is $P \sim 11$ days and its average apparent magnitude is $\sim (14.8 + 14.1)/2 \text{ mag}$, i.e. $m = 14.45 \text{ mag}$. (2 points)

[A careful student will notice that the graph is not *upside-down* symmetrical so he/she choose some value closer to the bottom; that is $m = 14.5 \text{ mag}$. (1 point)]

From Figure 1, we derive that for a period of 11 days the expected absolute magnitude of the cepheid is $M \approx -4.2$. (1 point)



[A careful student will notice that the graph is logarithmic so he/she choose a value closer to 4.3]

(1 point)

Using the formula $m - M = -5 + 5 \log r$, where r is the distance of the cepheid, we get

$$\log r = (14.45 + 4.2 + 5) / 5 = 4.73, \text{ then } r = 10^{4.73} \approx 57500 \text{ pc or } r = 57.5 \text{ kpc} \quad \textbf{(3 points)}$$

(b) Assuming $A = 0.25$, then $\log r = (14.45 + 4.2 + 5 + 0.25) / 5 = 4.78$, then $r = 10^{4.78} \sim 53000 \text{ pc or}$

$$r = 64.5 \text{ kpc.} \quad \textbf{(2 points)}$$

5. The optical spectrum of a galaxy, whose distance had been measured to be 41.67 Mpc, showed the Balmer H α line ($\lambda_o = 656.3 \text{ nm}$) redshifted to $\lambda = 662.9 \text{ nm}$. (a) Use this distance to calculate a value of the Hubble constant, H_o . (b) Using your results, estimate the Hubble time of the Universe.

$$\textbf{Answer: (a)} \quad z = \frac{\lambda - \lambda_o}{\lambda_o} = \frac{662.9 - 656.3}{656.3} \approx 0.01 \quad \textbf{(3 Point)}$$

This is small enough that we can use the classical equation for the expansion of the Universe.

$$H_o = \frac{cz}{r} = \frac{3 \times 10^5 \text{ km s}^{-1} \times 10^{-2}}{41.67 \text{ Mpc}} \quad \text{or } H_o = 72.4 \frac{\text{km s}^{-1}}{\text{Mpc}} \quad \textbf{(4 Points)}$$

$$\textbf{(b)} \quad t_H \approx \frac{1}{H_o} \quad \text{or } t_H = 13.5 \text{ Gyr} \quad \textbf{(3 Point)}$$

6. A star has an effective temperature $T_{eff} = 8700 \text{ K}$, absolute magnitude $M = 1.6$ and apparent magnitude $m = 7.2$. Find (a) the star's distance, r , (b) its luminosity, L , and (c) its radius, R . (Ignore extinction).

$$\textbf{Answer : (a)} \quad \text{Its distance is calculated from equation: } m - M = 5 \log(r) - 5, \text{ or } 7.2 - 1.6 + 5 = 5 \log(r) \rightarrow \log(r) = 2.12 \text{ and } r = 132 \text{ pc} \quad \textbf{(3 Points)}$$

$$\textbf{(b)} \quad \text{Its Luminosity is calculated from equation: } M_{\odot} - M = 2.5 \log\left(\frac{L}{L_{\odot}}\right) \text{ or}$$

$$4.8 - 1.6 = 2.5 \log\left(\frac{L}{L_{\odot}}\right) \text{ or } \log\left(\frac{L}{L_{\odot}}\right) = 1.28 \rightarrow \left(\frac{L}{L_{\odot}}\right) = 19.05 \text{ or } L = 19.15 \times L_{\odot} = 19.15 \times 3.9 \times 10^{33} \text{ and } L = 7.4 \times 10^{34} \text{ erg sec}^{-1} \quad \textbf{(4 Points)}$$

(c) Its radius can be easily calculated from equation: $L = 4\pi\sigma R^2 T_{eff}^4$, from which

$$R = \frac{1}{T_{eff}^2} \sqrt{\frac{L}{4\pi\sigma}} \text{ from which we get: } R = 1.35 \times 10^{11} \text{ cm} \quad \textbf{(3 Points)}$$

7. A star has visual apparent magnitude $m_V = 12.2$ mag, parallax $\pi = 0''.001$ and effective temperature $T_{eff} = 4000$ K. Its bolometric correction is $B.C. = -0.6$ mag. (a) Find its luminosity as a function of the solar luminosity. (b) What type of star is it? (i) a *red giant*? (ii) a *blue giant*? or (iii) a *red dwarf*? Please write (i), (ii) or (iii) in your answer sheet.

Answer : (a) First its bolometric magnitude is calculated from equation: $M_V - m_V = 5 - 5 \log(r)$ or equivalent: $M_V - m_V = 5 + 5 \log \pi \rightarrow M_V = 12.2 + 5 + 5 \log(0''.001) = 12.2 + 5 - 15 = 2.2$ mag. Its barycentric correction is: $B.C. = M_{bol} - M_V$ and $M_{bol} = B.C. + M_V$ or $M_{bol} = -0.6 + 2.2$ or $M_{bol} = 1.6$ mag .

(4 Points)

Then its Luminosity is calculated from:

$$M_{\odot} - M_{bol} = 2.5 \log \left(\frac{L}{L_{\odot}} \right), \text{ or } 4.72 - 1.6 = 2.5 \log \left(\frac{L}{L_{\odot}} \right) \text{ or } \log \left(\frac{L}{L_{\odot}} \right) = 1.25 \text{ and } L = 17.70 L_{\odot}$$

(2 Point)

(b) Type of star: A star with $M_{bol} = 1.6$ mag, $L = 17.7 L_{\odot}$ and $T_{eff} = 4000$ K is much brighter and much cooler than the Sun (see *Table of constants*). Therefore it is **(i)** a red giant star. **(4 Points)**

8. A binary system of stars consists of star (a) and star (b) with brightness ratio 2. The binary system is difficult to resolve and is observed from the Earth as one star of 5th magnitude. Find the apparent magnitude of each of the two stars (m_a, m_b).

Answer : The apparent magnitude of star (a) is m_a , of star (b) is m_b and that of the system as a whole is m_{a+b} . The corresponding apparent brightnesses are ℓ_a, ℓ_b and $\ell_{a+b} = \ell_a + \ell_b$.

For star (a) :

$$m_{a+b} - m_a = -2.5 \log \left(\frac{\ell_a + \ell_b}{\ell_a} \right) \text{ and because } \frac{\ell_b}{\ell_a} = \frac{1}{2}, \text{ we get } m_a = m_{a+b} + 2.5 \log (1 + \frac{1}{2}) \text{ or}$$

$$m_a = 5 + 2.5 \log (3/2) \text{ and finally } m_a = 5.44 \text{ mag} . \text{ Similarly for star (b):} \quad \textbf{(5 point)}$$

$$m_{a+b} - m_b = -2.5 \log \left(\frac{\ell_a + \ell_b}{\ell_b} \right) \text{ and because } \frac{\ell_a}{\ell_b} = 2, \text{ we get } m_b = m_{a+b} + 2.5 \log (3) \text{ or}$$

$$m_b = 5 + 2.5 \log (3) \text{ and finally } m_b = 6.19 \text{ mag.} \quad \textbf{(5 Point)}$$

9. Find the equatorial coordinates (*hour angle* and *declination*) of a star at Madrid, geographic latitude $\phi = 40^\circ$, when the star has zenith angle $z = 30^\circ$ and azimuth $A = 50^\circ$ (azimuth as measured from the South)

Answer : From the position triangle $\Pi Z_V \Sigma$ (Figure 3) of the star, Σ , we get, by using the *cosine law* for a spherical triangle:

$$\cos(90 - \delta) = \cos(90 - a) \times \cos(90 - \phi) + \sin(90 - a) \times \sin(90 - \phi) \times \cos(180 - A) \quad \textbf{(2 Point)}$$

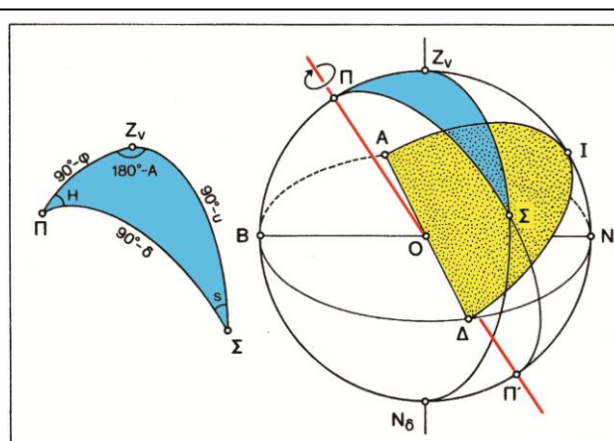


Figure 3. The position triangle



where δ is the star's *declination*, a its *altitude* ($a = 90^\circ - z$), φ the *geographical latitude* of the observer, H the stars *hour angle* and A the star's *azimuth*.

This can be written as:

$$\sin \delta = \cos z \times \sin \varphi - \sin z \times \cos \varphi \times \cos A \text{ or}$$

$$\sin \delta = \cos 30^\circ \times \sin 40^\circ - \sin 30^\circ \times \cos 40^\circ \times \cos 50^\circ \quad (2 \text{ Point})$$

or

$$\sin \delta = 0.866 \times 0.643 - 0.500 \times 0.766 \times 0.643 = 0.311. \quad \delta = 18^\circ 05' \quad (1 \text{ Point})$$

Using the sine law for the spherical triangle, we get:

$$\frac{\sin H}{\sin(90 - a)} = \frac{\sin(180^\circ - A)}{\sin(90^\circ - \delta)} \text{ or } \frac{\sin H}{\sin z} = \frac{\sin A}{\cos \delta} \quad (2 \text{ Point})$$

$$\rightarrow \sin H = \sin 50^\circ \times \frac{\sin 30^\circ}{\cos(18^\circ 07')} = \frac{0.766 \times 0.5}{0.950} \quad (2 \text{ Point})$$

$$\text{or } \sin H = 0.403. \text{ Therefore: } H = 23^\circ 46' \text{ or } H = 1^h 35^m 03^s. \quad (1 \text{ Point})$$

10. In the centre of our Galaxy, in the intense radio source Sgr A*, there is a black hole with large mass. A team of astronomers measured the angular distance of a star from Sgr A* and its orbital period around it. The maximum angular distance was $0.12''$ (arcsec) and the period was 15 years. Calculate the mass of the black hole in solar masses, assuming a circular orbit.

$$\text{Answer: } F = -\frac{GM_{BH}M_*}{R^2} = -\frac{M_*v^2}{R} \quad (2 \text{ Point})$$

$$\text{But } v = \frac{2\pi R}{P}. \text{ Therefore } \frac{GM_{BH}}{R^2} = 4\pi^2 \frac{R}{P^2} \text{ or } GM_{BH} = 4\pi^2 \frac{R^3}{P^2} \quad (2 \text{ Points})$$

Similarly:

$$GM_\odot = 4\pi^2 \frac{(1AU)^3}{(1yr)^2} \quad \text{h} \quad G = \frac{1}{M_\odot} 4\pi^2 \frac{(1AU)^3}{(1yr)^2} \quad (2 \text{ Point})$$

$$\text{From Kepler's 3rd law we get: } \frac{M_{BH}}{M_\odot} = \frac{(R/1AU)^3}{(P/1yr)^2} \quad (1 \text{ Point})$$

Inserting the given data we find the distance of the star from the black hole:

$$R = \frac{0.12}{200,000} (8000)(3 \times 10^{18} \text{ cm}) = 1.4 \times 10^{16} \text{ cm} = 960 \text{ AU} \quad (2 \text{ Point})$$

$$\text{Therefore: } \frac{M_{BH}}{M_\odot} = \frac{(960)^3}{(15)^2} = 4 \times 10^6 \quad \text{from which we calculate the mass of the black hole:}$$

$$M_{BH} = 4 \times 10^6 M_\odot \quad (1 \text{ Point})$$

11. What is the maximum altitude, a_M (max), at which the Full Moon can be observed from Thessaloniki? The geographical latitude of Thessaloniki is $\varphi_\Theta = 40^\circ 37'$. Take into account as many factors as possible.



Answer: In order to have Full Moon, the Moon should be diametrically opposite the Sun, i.e. the three bodies, Sun – Earth – Moon should be on a straight line. If the orbital plane of the Moon coincided with the ecliptic, the maximum altitude of the Full Moon would be $90^\circ - \varphi_\oplus + 23.5^\circ$.

(2 point).

Because the orbital plane of the Moon is inclined by 5.14° ($5^\circ 18'$) to the plane of the ecliptic, the maximum angle is larger: $90^\circ - 40.6^\circ + 23.5^\circ + 5.3^\circ$, or (P)

$$\alpha_M(\text{max}) = 79.8^\circ \quad (3 \text{ points})$$

Geocentric parallax of Moon for this situation is 0.33° , (3 points)

whereas refraction is only $0.2'$. (1 point)

Final answer is therefore: $79.8^\circ - 0.3^\circ = 79.5^\circ$ (1 point)

12. *Sirius A*, with visual magnitude $m_V = -1.47$ (the brighter star on the sky) and with stellar radius $R_A = 1.7R_\odot$, is the primary star of a binary system. The existence of its companion, *Sirius B*, was deduced from astrometry in 1844 by the well known mathematician and astronomer Friedrich Bessel, before it was directly observed. Assuming that both stars were of the same spectral type and that *Sirius B* is fainter by 10 mags ($\Delta m = 10$), calculate the radius of *Sirius B*.

Answer: The distance of the two stars from our solar system is the same. Therefore

$$m_B - m_A = 2.5 \log \frac{\frac{L_A}{4\pi r^2}}{\frac{L_B}{4\pi r^2}} = 2.5 \log \frac{L_A}{L_B} \quad (5 \text{ Point})$$

From which we get $L_A = 10^4 L_B$. From equation $L = 4\pi R^2 \sigma T_{\text{eff}}^4$ we get

$$L_A/L_B = (R_A/R_B)^2 (T_A/T_B)^4$$

Assuming that the two stars belong to the same spectral type (and therefore $T_A = T_B$) we get

$$R_B = 0.01R_A = 0.01 \times 1.7 \times 696000 \text{ km or } R_B = 1.2 \cdot 10^4 \text{ km} \quad (5 \text{ Point})$$

13. Recently in London, because of a very thick layer of fog, the visual magnitude of the Sun, became equal to the (usual – as observed during cloudless nights) magnitude of the full Moon. Assuming that the reduction of the intensity of light due to the fog is given by an exponential equation, calculate the exponential coefficient, τ , which is usually called *optical depth*.

Answer: The absorption due to the fog in London is obviously

$$A = -26.8 - (-12.74) = -14.06 \text{ mag.} \quad (2 \text{ Point})$$

$$\text{Rearranging the equation } I_\nu(r) = I_\nu(0) \times e^{-\tau}, \text{ we get } \frac{I_\nu(0)}{I_\nu(r)} = e^\tau, \quad (5 \text{ Points})$$

or

$$A = \Delta m = 2.5 \times \log(e) \times (-\tau) \text{ or } \tau = (-A)/(2.5 \times \log e) = 14.06/1.08 \text{ } \tau = 12.9 \quad (3 \text{ Points})$$



14. What is the hour angle, H , and the zenith angle, z , of *Vega* ($d = 38^\circ 47'$) in Thessaloniki ($\lambda_1 = 1^{\text{h}}32^{\text{m}}$, $\phi_1 = 40^\circ 37'$), at the moment it culminates at the local meridian of Lisbon ($\lambda_2 = -0^{\text{h}}36^{\text{m}}$, $\phi_2 = +39^\circ 43'$)?

Answer: By definition at the moment when the star culminates in Lisbon, its hour angle is exactly 0° . Therefore its hour angle in Thessaloniki is $0^\circ + (\lambda_1 - \lambda_2)$ or $H = 02^{\text{h}}08^{\text{m}}$. **(3 Points)**

Using the cosine law equation $\cos z = \cos(90-\phi) \times \cos(90-d) + \sin(90-\phi) \times \sin(90-d) \times \cos H$, **(4 Points)**

the zenith distance at Thessaloniki can be calculated to $24^\circ 33'$ **(3 Points)**

15. The Doppler shift of three remote galaxies has been measured with the help of Spectral observations:

Galaxy	Redshift, z
3C 279	0.536
3C 245	1.029
4C41.17	3.8

- (a) Calculate their apparent recession velocity (1) using the classical approach, (2) using the approximate formula $v = c \ln(1+z)$, that is often used by cosmologists and (3) using the special relativistic approach.
(b) For all three formulae, at what percentage of the speed of light do they appear to recede?
(c) Which of (1) classical, (2) special relativity (3) approximate cosmological.

Answer: The recession velocity is calculated by either the classical relation, $v_c = z \times c$, or the relativistic relation, $v_r = \frac{(1+z)^2 - 1}{(1+z)^2 + 1} c$. The calculations for the three galaxies are summarized in the following Table:

Galaxy	v_c (km/s)	V_a (km/s)	v_r (km/s)	$v_r/c \times 100$
3C279	160800	128750	121390	40%
3C245	308700	212260	182740	61%
4C41.17	1140000	470580	275040	92%

- (1) Table 1, Columns 2,3,4 **(5 Points)**
(2) Column 5 **(3 Points)**
(3) If the student answers “Classical” **(0 Points)**
If the student answers “Special Relativity” **(1 Point)**
If the student answers “Approximate formula” **(2 Points)**



Theoretical Exam - Long Questions

Question 1

In a homogeneous and isotropic universe, the matter (baryonic matter + dark matter) density parameter $\Omega_m = \frac{\rho_m}{\rho_c} = 32\%$, where ρ_m is the matter density and ρ_c is the critical density of the Universe.

- (1) Calculate the average matter density in our local neighbourhood.
- (2) Calculate the escape velocity of a galaxy 100 Mpc away from us. Assume that the recession velocity of galaxies in Hubble's law equals the corresponding escape velocity at that distance, for the critical density of the Universe that we observe.
- (3) The particular galaxy is orbiting around the centre of our cluster of galaxies on a circular orbit. What is the angular velocity of this galaxy on the sky?
- (4) Will we ever discriminate two such galaxies that are initially at the same line of sight, if they are both moving on circular orbits but at different radii (answer “Yes” or “No”)? [Assume that the Earth is located at the centre of our local cluster.]

Answer:

(1) The critical density $\rho_c = \frac{3H_0^2}{8\pi G}$ (9 points)

If the matter density parameter $\Omega_m = \frac{\rho_m}{\rho_c}$ is 32% , thus $\rho_m = 0.32 \frac{3H_0^2}{8\pi G}$. (3 points)

From the latest estimate of $H_0 = 67.8 \frac{\text{km s}^{-1}}{\text{Mpc}}$ we obtain $\rho_m = 8.6 \times 10^{-27} \text{ kg m}^{-3}$ (4 points)

(2) The escape velocity is $v_{\text{esc}} = \sqrt{\frac{2GM}{d}}$. (4 points)

By replacing $M = \rho_c \frac{4\pi}{3} d^3$ (2 points)

we obtain, for the escape velocity within $d = 100 \text{ Mpc}$

$$v_{\text{esc}} = \sqrt{\frac{8\pi}{3} G \times 0.32 \times \frac{3H_0^2}{8\pi G}} \times 100 \text{ Mpc} = 3835 \text{ km s}^{-1} \quad \text{(8 points)}$$

(3) If a galaxy is orbiting around the centre of our galaxy, its velocity is $\frac{1}{\sqrt{2}}$ of its escape velocity. Thus

$$\omega = \frac{v}{d} = \frac{v_{\text{esc}}}{d} \cdot \frac{1}{\sqrt{2}} = \frac{H_0 d \sqrt{\Omega_m / 2}}{d} = \frac{\sqrt{0.32} (67.8 \times 10^3 \text{ m/s}) / (3.09 \times 10^{22} \text{ m})}{\sqrt{2}} = 8.8 \times 10^{-19} \text{ rad/s} \quad \text{(12 Points)}$$

This is

$$1.8 \times 10^{-13} \text{ arcsec/s} \quad \text{and it does not depend on the distance } d.$$

(4) Therefore we will never be able to resolve them and the answer is “No”. (8 Points)



Question 2

A spacecraft is orbiting the Near Earth Asteroid (2608) *Seneca* (staying continuously very close to the asteroid), transmitting pulsed data to the *Earth*. Due to the relative motion of the two bodies (the asteroid and the *Earth*) around the *Sun*, the time it takes for a pulse to arrive at the ground station varies approximately between 2 and 39 minutes. The orbits of the *Earth* and *Seneca* are coplanar. Assuming that the *Earth* moves around the *Sun* on a circular orbit (with radius $a_{\text{Earth}} = 1$ AU and period $T_{\text{Earth}} = 1$ yr) and that the orbit of *Seneca* does not intersect the orbit of the *Earth*, calculate:

- (1) the semi-major axis, a_{Sen} the eccentricity, e_{Sen} of *Seneca's* orbit around the *Sun*
- (2) the period of *Seneca's* orbit, T_{Sen} and the average period between two consecutive oppositions, T_{syn} of the *Earth-Seneca* couple
- (3) an approximate value for the mass of the planet Jupiter, M_{Jup} (assuming this is the only planet of our Solar system with non-negligible mass compared to the *Sun*). Assume that the presence of Jupiter does not influence the orbit of *Seneca*.

Answer :

- (1) For $\Delta t_b = 2$ min = 120 sec, the distance travelled by a light pulse is $R_1 = c \times \Delta t$ or $R_1 = 0.24$ AU, while for $\Delta t_a = 39$ min the maximum distance is $R_2 = c \times \Delta t$ or $R_2 = 4.67$ AU. **(4 points)**

Since the orbits do not intersect and the R_2 exceeds by far 1 AU, the orbit of *Seneca* is exterior to that of the *Earth*. R_1 corresponds to the minimum relative distance of the two bodies (i.e. at opposition), while R_2 corresponds to the maximum relative distance (i.e. at conjunction).

If $q = a \times (1 - e)$ is the perihelion and $Q = a \times (1 + e)$ the aphelion distance of *Seneca*, then $R_1 = q - 1$ AU (the minimum distance of *Seneca* from the *Sun* minus the semi-major axis of the *Earth's* orbit), while $R_2 = Q + 1$ AU (the maximum distance of *Seneca* from the *Sun* plus the semi-major axis of the *Earth's* orbit). **(6 points)**

Thus,

$$a_{\text{Sen}} (1 - e_{\text{Sen}}) = 1 + R_1 = 1.24 \text{ AU}$$

$$a_{\text{Sen}} (1 + e_{\text{Sen}}) = R_2 - 1 = 3.67 \text{ AU}$$

$$\text{from which one finds } a_{\text{Sen}} \sim 2.46 \text{ AU}$$

$$\text{and } e_{\text{Sen}} \sim 0.49$$

(2 point)

(2 points)

(figures rounded to 2 dec. digits)

- (2) The period, T_{Sen} , of *Seneca's* orbit can be found by using Kepler's 3rd law for *Seneca* and the *Earth* (ignoring their small masses, compared to the *Sun's*):

$$a_{\text{Sen}}^3 / T_{\text{Sen}}^2 = a_{\text{Earth}}^3 / T_{\text{Earth}}^2 = 1$$

from which one finds $T_{\text{Sen}} \sim 3.87$ yr

(6 points)

(2 point)

(Alternatively one can use Kepler's law for *Seneca* only and use natural units for the mass of the *Sun* and G , i.e. [mass] = 1 M_{Sun} , [t] = 1 yr and [r] = 1 AU, in which case $T_{\text{Sen}} = a_{\text{Sen}}^{3/2} = 3.86$ yr)

(8 points)

Assuming non-retrograde orbit, the synodic period, T_{syn} of *Seneca* and *Earth* is given by



$$1/T_{\text{syn}} = 1/T_{\text{Earth}} - 1/T_{\text{Sen}} \text{ (Seneca is superior)}$$

(6 points)

$$\text{which gives } T_{\text{syn}} = 1.35 \text{ yr}$$

(2 point)

(3) From Kepler's law, we can calculate the mass of the *Sun*

$$4\pi^2 a_{\text{Sen}}^3 / T_{\text{Sen}}^2 = G M_{\text{Sun}}, \text{ i.e. } M_{\text{Sun}} \sim 1.988 \times 10^{30} \text{ kg} \quad (\text{depending on accuracy}) \quad (4 \text{ points})$$

Then, using Kepler's 3rd law for *Seneca* and *Jupiter*, one gets

$$(a_{\text{Jup}}^3 / T_{\text{Jup}}^2) / (a_{\text{Sen}}^3 / T_{\text{Sen}}^2) = (M_{\text{Sun}} + M_{\text{Jup}}) / M_{\text{Sun}} = 1 + x \quad (4 \text{ points})$$

where $x =$ is the mass ratio of *Jupiter* to the *Sun*. Then, solving for x we get (depending on the accuracy used in determining the elements of *Seneca*'s orbit)

$$x = 0.0016, \text{ thus } M_{\text{Jup}} = 3.2 \times 10^{27} \text{ kg} \quad (4 \text{ points})$$

Question 3

(1) Using the *virial theorem* for an isolated, spherical system, i.e, that $-2\langle K \rangle = \langle U \rangle$, where “ K ” is the average kinetic energy and “ U ” is the average potential energy of the system, determine an expression for the total mass of a cluster of galaxies if we know the radial velocity dispersion, σ , of the cluster's galaxy members and the cluster's radius, R . Assume that the cluster is isolated, spherical, has a homogeneous density and that it consists of galaxies of equal mass.

(2) Find the *virial mass*, i.e. the mass calculated from the *virial theorem*, of the Coma cluster, which lies at a distance of 90 Mpc from us, if you know that the radial velocity dispersion of its member galaxies is $\sigma_{v_r} = 1000 \text{ km/s}$ and that its angular diameter (on the sky) is about 4° .

(3) From observations, the total luminosity of the galaxies comprising the cluster is approximately $L = 5 \times 10^{12} L_\odot$. If the mass to luminosity ratio, M/L , of the cluster is ~ 1 (assume that all the mass of the cluster is visible mass), this should correspond to a total mass $M \sim 5 \times 10^{12} M_\odot$ for the mass of the cluster. Give the ratio of the luminous mass to the total mass of the cluster you derived in question (2).

Answer:

(1) Using the virial theorem for our isolated, spherical system of N galaxies of mass m , each, we get

$$-2\langle K \rangle = \langle U \rangle \Rightarrow -\frac{2}{N} \sum_1^N \frac{1}{2} m_i u_i^2 = \langle U \rangle \Rightarrow -\frac{m}{N} \sum_1^N u_i^2 = \frac{U}{N} \quad (3 \text{ point})$$

where $\frac{1}{N} \sum_1^N u_i^2 \approx \langle u^2 \rangle = \langle u_r^2 \rangle + \langle u_\theta^2 \rangle + \langle u_\phi^2 \rangle$, where u_r , u_θ and u_ϕ are the radial velocity and the two perpendicular velocities on the plane of the sky of the members of the cluster.

(6 points)



Assuming that $\langle u_r^2 \rangle \sim \langle u_\theta^2 \rangle \sim \langle u_\phi^2 \rangle$, we have

$$-3m\langle u_r^2 \rangle = U / N = \left(-\frac{3}{5} \frac{GM^2}{R}\right) / N = -\frac{3}{5} \frac{GMm}{R} \Rightarrow M \approx \frac{5R\langle u_r^2 \rangle}{G} \quad (13 \text{ points})$$

(where we used the gravitational potential of a spherical homogeneous mass M enclosed within radius R)

Alternatively the student can give a rougher order of magnitude estimate

$$-2\langle K \rangle = \langle U \rangle \Rightarrow -2 \cdot \frac{1}{2} \langle u_r^2 \rangle \approx -\frac{GM}{R} \text{ etc} \quad \text{If they do not use the exact formula:} \quad (11 \text{ Points})$$

(2) From the result of the previous question we have $M \approx \frac{5R\langle u_r^2 \rangle}{G}$, where

$$\sqrt{\langle u_r^2 \rangle} = \sigma = 1000 \text{ km s}^{-1} \quad (8 \text{ points})$$

The angular diameter of the cluster is $\phi = 4^\circ$ at a distance of $d = 90 \text{ Mpc}$. Therefore the diameter D of the cluster in Mpc is calculated from:

$$\tan \phi = \frac{D}{d} \Rightarrow \phi(\text{rad}) \approx \frac{D}{d} \Rightarrow D \approx 4 \times \frac{\pi}{180} \times 90 \text{ Mpc} \approx 6.3 \text{ Mpc} \Rightarrow R \approx 3 \text{ Mpc} \quad (11 \text{ points})$$

$$\text{Therefore } M \approx \frac{5R\langle u_r^2 \rangle}{G} = 3.6 \times 10^{15} M_\odot \quad (3 \text{ point})$$

(3) $\frac{M_{\text{virial}}}{L_{\text{galaxies}}} = \frac{3.6 \times 10^{15} M_\odot}{5 \times 10^{12} L_\odot} = 720 \frac{M}{L}$ This is obviously much larger from $\frac{M_\odot}{L_\odot} = 1$ which is found from the visible mass of the cluster. Thus $\frac{M}{M_{\text{virial}}} = \frac{1}{720}$. (6 points)

OBSERVATION



7th International Olympiad
on Astronomy & Astrophysics

VOLOS 2013 HELLAS



Observational Test

Question 1.

Find the field of view of the telescope with the eyepiece provided by the attendant.

Answer: The attendant will provide a chronometer.

The student should select and observe any star from the “*Bright Stars catalogue visible from Greece*” that is provided by the attendant. The name of the star and its declination is written on the data sheet.

(1 point)

(Note: If the student selects α UMi the attendant should not warn him).

The student measures and writes down the *crossing time* of the star that he/she has selected.

(Marking scheme: ± 4 s: 100%, ± 6 s: 80%, ± 8 s: 60%, ± 12 s: 40%, ≥ 12 s: 0).

(5 points)

The field of view is then calculated by the student on the spot, by using the formula:

$$FoV = \omega \times t \times \cos(dec) = \frac{360^\circ}{23^h 56^m 4^s.1} \times t \times \cos(dec) \quad (4 \text{ points})$$

(4 points)

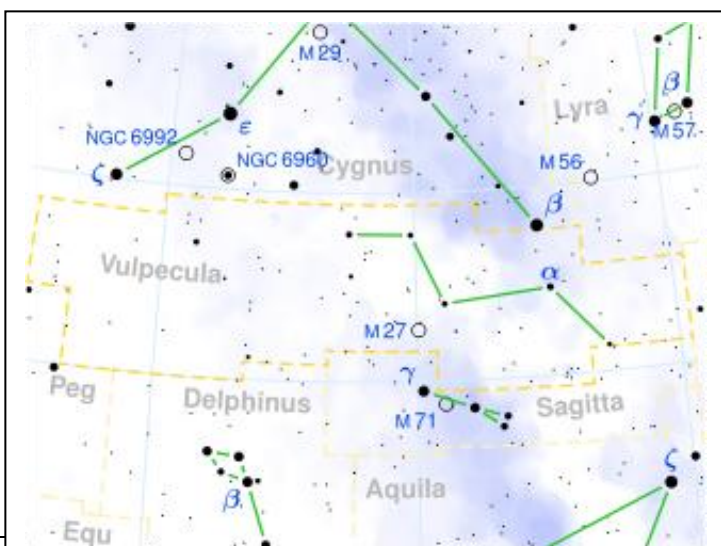
(Example: for Capella [$dec = 46^\circ.0$, $\cos(46.0) = 0.6947$] and transit time $t = 3^m 31^s = 3.53$ min

we get: $FoV = \frac{(360 \times 24)'}{1436.001 \text{ min}} \times 3.53 \text{ min} \times 0.6947 = 36'.9$)

[Maximum allowed time **10 minutes**]

Question 2.

Locate the bright star γ *Sagitta* ($RA = 19^h 58^m 45.39^s$, $Dec = +19^\circ 29' 31.5''$), which lies between the constellations of Lyra and Delphinus. Then aim and locate the famous Dumbbell Nebula, M27 ($RA = 19^h 59^m 36.34^s$, $+22^\circ 43' 16.09''$) in the center of the field of view. The observing spot is rather dark and you cannot read the setting circles!



The field in the vicinity of the Dumbbell Nebula

Answer: The student should recognize the constellation of Sagitta and tell the attendant to which direction is the “head of the arrow”

(2 point)

Then points the telescope at γ *Sagitta*, which is the brightest star of the constellation.

(3 point)

The students should notice that the two targets have very similar *Right Ascension*. Therefore given the *RA* and *Dec* of the bright star γ *Sagitta* ($m =$



3.51 mag), should be able to quickly locate *M27* as the equatorial mounting is already aligned. Then should keep the *RA* axis locked, release the Dec knob and turn the telescope by about 3.25 degrees towards Polaris. The Dumbell Nebula will appear in the field of view.

(5 point)

[Marking scheme: Points given according to time spent. Total time required: ≤ 4 min: 100%, ≤ 5 min: 80%, ≤ 6 min: 50%.

[Maximum allowed time **6 minutes**]

(Note: If the student fails to locate *M27* and complains, the attendant does it 30 s)

Question 3:

At 14 o'clock local time in the morning of the spring equinox a rare transit of Mercury is going to take place. A team of astronomers reaches a mountain top, early in the morning, in order to align his telescope and then observe the transit. The site is new and they do not know the geographical coordinates. Unfortunately the sky is covered with clouds. No stars are visible. The telescope cannot be aligned. The sky is overcast until 11 o'clock. The Sun becomes visible. An experienced astronomer manages to roughly align the telescope in less than 2 minutes! He only uses a water bubble.

You are given the telescope of the 7th IOAA and a water level. Assume that it is spring equinox and that the time is 12 o'clock. A fake Sun is shining. Could you align the telescope?

(Note: Obviously for this exercise, a telescope tube is not necessary, therefore, for the sake of convenience, the telescope will be equipped with a rough paper-tube and without counter weights.

Answer: First the student levels the tripod with the help of the water bubble. Then he/she adapts the equatorial mount on the tripod. Because it is spring equinox, the declination of the Sun is 0° . At this point the student should immediately set the Declination circle of the telescope at 0° and secures the break knob. The declination axis is calibrated. Then he/she rotates the *RA* axis and, by using the water bubble, makes the tube of the telescope horizontal (pointing toward the East). He rotates the *RA* setting circle to show 0 hours. Then he/she rotates again the *RA* axis until the setting circle shows 6 hours. Obviously, if the azimuth axis had been correctly set, at this point the telescope should be pointing somewhere on the local meridian. Then, exactly at 12 o'clock, when the sun crosses the meridian, he turns the azimuthal axis of the telescope until he observes the Sun above or below the direction where the telescope is pointing. Now the telescope is pointing toward the local meridian. Finally he turns the latitude (altitude) axis of the telescope, up or down, until he aims the sun. The polar axis of the telescope is immediately aligned!

(Note: This is an indoors exercise). [Maximum allowed time **16 minutes**]



Observational test

Question:

At 14 o'clock local time in the morning of the spring equinox a rare transit of Mercury is going to take place. A team of astronomers reaches a mountain top, early in the morning, in order to align his telescope and then observe the transit. The site is new and they do not know the geographical coordinates. Unfortunately the sky is covered with clouds. No stars are visible. The telescope cannot be aligned. The sky is overcast until 11 o'clock. The Sun becomes visible. An experienced astronomer manages to roughly align the telescope in less than 2 minutes! He only uses a water bubble.

You are given the telescope of the 7th IOAA and a water level. Assume that it is spring equinox and that the time is 12 o'clock. A fake Sun is shining. Could you align the telescope?

(Note: Obviously for this exercise, a telescope tube is not necessary, therefore, for the sake of convenience, the telescope will be equipped with a rough paper-tube and without counter weights.

DATA ANALYSIS



7th International Olympiad
on Astronomy & Astrophysics
VOLOS 2013 HELLAS

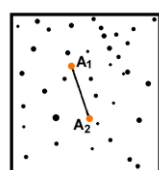
Data analysis

Question 1.

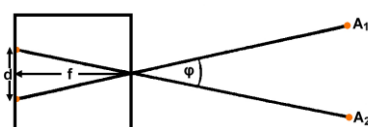
In Figure 1, part of the constellation of Ursa Major is shown. It was taken with a digital camera with a large CCD chip ($17\text{mm} \times 22\text{mm}$). Find the focal length, f , of the optical system and give the error of your results.



Figure 1. Part of the constellation of Ursa Major.



(a)



(b)

Figure 1A. Calculating the distance scale.

Answer: Calculate the angular distance, φ , (in degrees or minutes of arc), of two bright stars, whose coordinates are given in the *List of Bright stars*. Preferably these stars should be chosen to be far apart (e.g. α UMa [$a_1 = 11^{\text{h}}03^{\text{m}}$, $\delta_1 = +61^\circ45'$] and η UMa [$a_2 = 13^{\text{h}}48^{\text{m}}$, $\delta_2 = +49^\circ19'$]).

- In order to calculate the angular distance of the two stars, the coordinates should be converted to decimal degrees (e.g. α UMa [$a_1 = 165^\circ.75$, $\delta_1 = +61.75^\circ$] and η UMa [$a_2 = 207.0^\circ$, $\delta_2 = +49^\circ.32$]) (2.5 Point)
- Use the *cosine law* to calculate the angular distance between the two stars:

$$\varphi = \arccos(\sin \delta_1 \times \sin \delta_2 + \cos \delta_1 \times \cos \delta_2 \times \cos (a_1 - a_2)) \rightarrow \varphi = 25^\circ.8448$$
 (10 Points)
- Measure the distance, d_0 , of the two stars in mm in Figure 1. $d_0 = 138$ mm. (2.5 Points)
- The photograph in Figure 1 does not have the same dimensions as the original photograph. Measure the length, ℓ , of the photograph in mm. $\ell = 140$ mm. This length corresponds to 22 mm. Convert d_0 to the distance, d , of the original photograph, $d = 138 \times \frac{22}{140}$ mm. $d = 21.6857$ mm (1.5 Points)
- The *image scale* $\left(\frac{\varphi}{d}\right)$ of the original photograph is $\frac{\varphi}{d} = \frac{25^\circ.8448}{21.6857} = 1.1918 \frac{\text{deg}}{\text{mm}}$ (2.5 Points)



5. The focal length is given from equation (see Figure 2): $\tan \frac{\varphi}{2} = \frac{d}{2f}$ or $f = \frac{d}{2 \tan \frac{\varphi}{2}}$ from

which the focal length is calculated: $f = \frac{21.6857}{2 \times 0.2294}$ or $f = 47.3 \text{ mm}$ (1.5 Points)

(If errors are included) (4 Points)

Question 2.

You are given 5 recent photographs of the solar photosphere shot at exactly the same time every two days (May 1 – May 9, 2013) in equatorial coordinates. You are also given two transparent Stonyhurst grids, which display heliocentric coordinates (heliocentric longitude, $\ell_{\odot}$, and heliocentric latitude, $b_{\odot}$). They cover the interval between April 28 to May 15. As the Earth does not orbit exactly around the Sun's equator, so, through the year, the solar equator seems to move up and down a little more than 7 degrees from the centre of the solar disc. This angle, B_0 , varies sinusoidally through the year. Furthermore, the axis of rotation of the Sun, as seen from the Earth, does not coincide with the axis of rotation of the Earth. The angle on the plane of the sky between the two axes, P_0 , also varies through the year. The numerical value of these angles (B_0 and P_0) are indicated on each of the 5 image of the Sun.

(1) Mark the axis of rotation of the Sun on each photograph.

(2) Choose 3 prominent sunspots that can be followed in all (or most) photographs and mark them as S1, S2 and S3 on the photos. Using the appropriate Stonyhurst grids, find their coordinates ($\ell_{\odot}, b_{\odot}$) for every day (May 1 to May 9) and note them down in Table 1.

Table 1

Date	Sunspot S1		Sunspot S2		Sunspot S3	
	$\ell_{\odot}$	$b_{\odot}$	$\ell_{\odot}$	$b_{\odot}$	$\ell_{\odot}$	$b_{\odot}$
May 1						
May 3						
May 5						
May 7						
May 9						

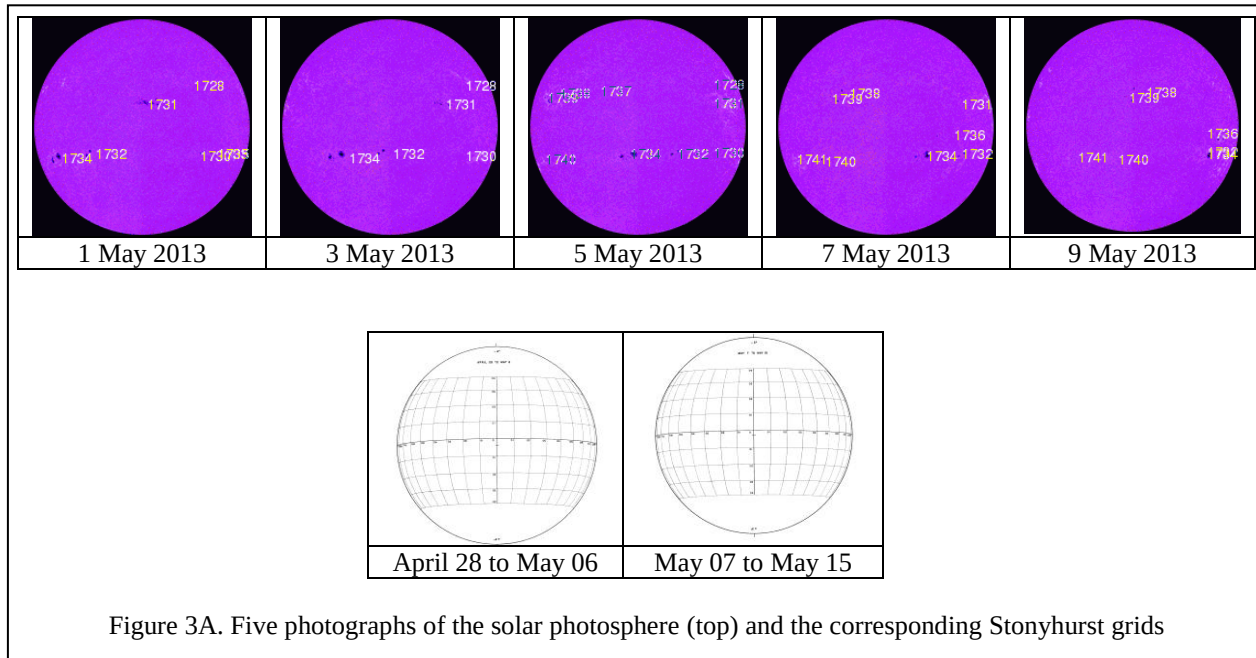
(3) Construct the diagrams $\Delta \ell_{\odot} / \Delta t$ for each sunspot.

(4) Calculate its synodic period (P) of rotation **in days** for each sunspot. Write down the result for each sunspot, P_{S1}, P_{S2}, P_{S3} .

(5) Calculate the average synodic period ($P_{\odot}$) of rotation of the Sun in days.



Answer: Five large photographs of the solar photosphere (adapted from www.spaceweather.com) and the appropriate (and on-scale) Stonyhurst grids will be given to the students (Figure 3A). The photographs given to the students, will not be annotated. Each photograph is accompanied by the angle B_0 and P_0 of the day of the observation. The student should calculate the rotation of the Sun by measuring the coordinates of at least 3 well recognized sunspots as they follow the rotation of the Sun.



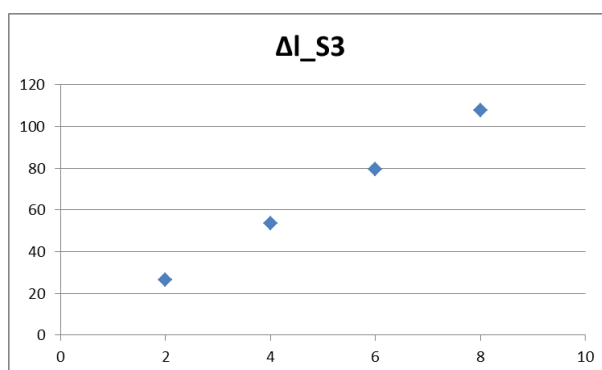
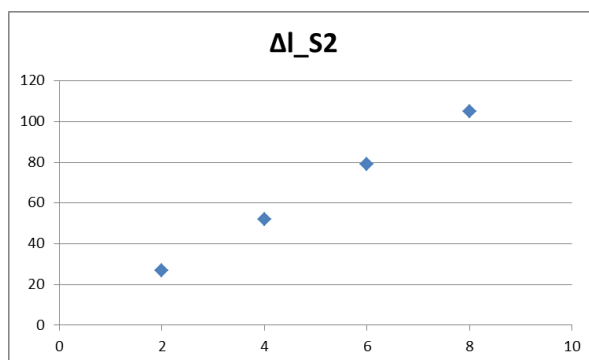
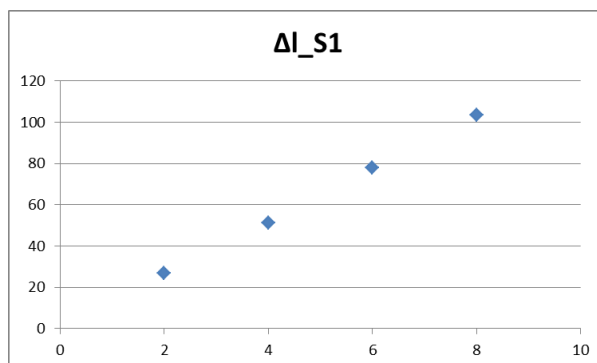
1. Correctly draw the axis of rotation of the Sun. This can be done by drawing a straight line at an angle of P_0 degrees anticlockwise from the vertical for each photograph (note: the photographs are given in equatorial coordinates.) **(4 Points)**
2. Choose the correct Stonyhurst grid and place it on each photograph, so that the solar axis on the grid coincides with the solar axis of the photograph. Estimate the heliocentric longitude, ($\ell_{\odot}$), and heliocentric latitude ($b_{\odot}$) with the help of the grid. Write down these coordinates in Table 1 for each of the 5 photographs.

Date	Sunspot S1		Sunspot S2		Sunspot S3	
	$\ell_{\odot}$	$b_{\odot}$	$\ell_{\odot}$	$b_{\odot}$	$\ell_{\odot}$	$b_{\odot}$
May 1	54	17.5	33	18.5	30.5	16
May 3	27	18	6	19	4	16
May 5	3	18.5	-19	18.5	-23	17
May 7	-24	18	-46	18	-49	16
May 9	-49.5	19	-72	18	-77	17

(12 Points)



3. Construct the diagram $\Delta\ell_{\odot}/\Delta t$ for each sunspot:



(2 for each graph)

(6 Points)

4. Calculate the rotation ($\Delta\ell_{\odot}/\Delta t$) of the Sun for each sunspot.

$$P_i = 8 * \frac{360}{\Delta\ell_{\odot i}}, \quad P_1 = 27.8 \text{ days}, \quad P_2 = 27.4 \text{ days}, \quad P_3 = 26.8 \text{ days}$$

(3 for each graph)

(9 Points)

5. Calculate the average period of the Sun

$$P_O = (P_1 + P_2 + P_3)/3 = 27.3 \text{ days} \quad 27.3 \text{ days}$$

(1.5 Point)



Question 3.

Figure 2 shows a photograph of the sky in the vicinity of the Hyades open cluster. The V-filter in the Johnson's photometric system was used. Figure 3 is a chart of the region with known V-magnitudes (m_V) of several stars (note that in order to avoid confusion with the stars, no decimal point is used, i.e. a magnitude $m_V = 8.1$ is noted as "81"). Hint: some of the stars may not be in the chart.

(a) Identify as many of the *stars shown with a number and an arrow* in Figure 3 and mark them on Figure 2.

(b) Comparing the V-magnitudes of the known stars in Figure 2, estimate the V-magnitudes of the *stars shown with a number and an arrow* in Figure 3.

Answer: The human eye can easily recognize differences of the order of $\Delta m_V = 0.1 - 0.2$ mag. The student should first align the photograph with the chart, which has a different scale and orientation. Then the magnitudes of the stars can easily be recognized. Note: not all arrowed stars can be found in the photograph. (0.5 Point for each star that has been correctly identified and 0.5 point for every correct magnitude within ± 0.3 magnitude) **(18 Points)**

Star	3	4	5	6	7	8	9	10	11
m	7.3 – 7.9	5.3 – 5.9	5.2 – 5.8	4.6 – 5.2	5.2 – 5.8	4.5 – 5.1	4.3 – 4.9	4.9 5.5	5.3 – 5.9

Stars 1, 2 and 12 are outside the boundaries of the photograph

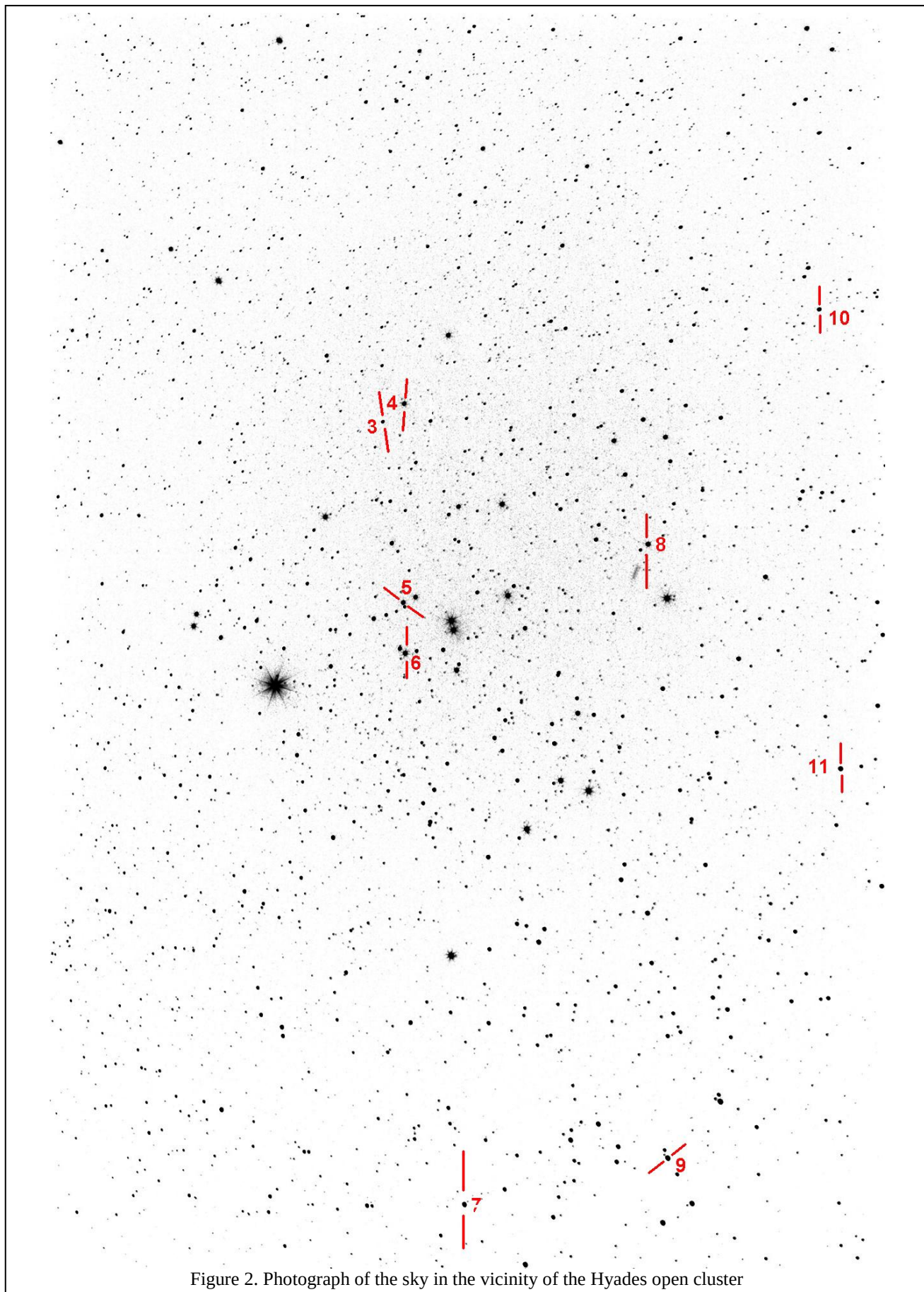


Figure 2. Photograph of the sky in the vicinity of the Hyades open cluster

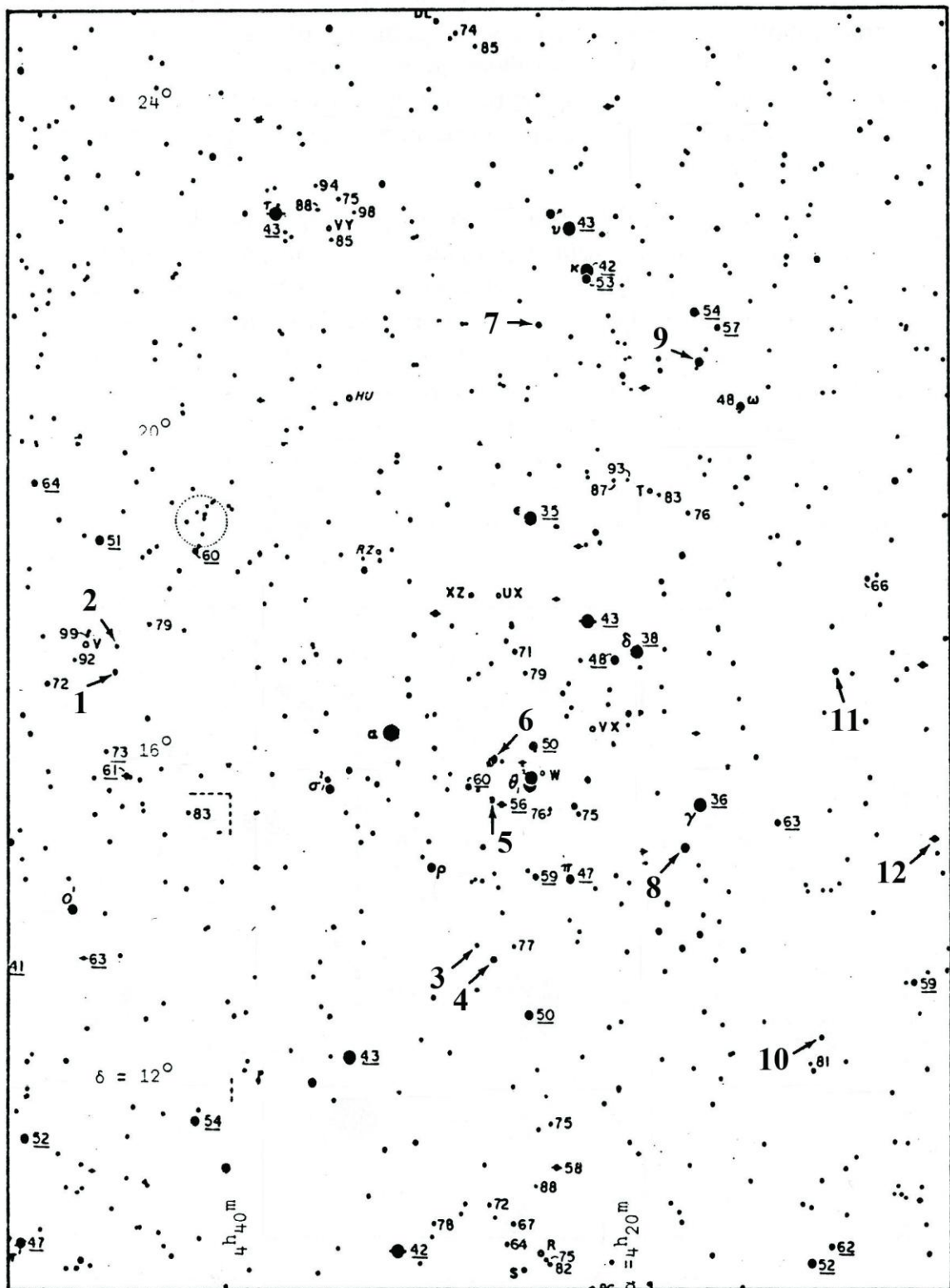


Figure 3. Sky chart of the region in the vicinity of the Hyades open cluster



Question 4.

Calculate the distance of the Hyades cluster using the *moving cluster* method (Figure 5).

1. In a *Text* file (*Hyades-stars.txt*) you are given a list of 35 stars from the field of the Hyades open cluster, observed by the Hipparcos space telescope.

The information listed in the columns of the text file for each of the 35 stars is: (a) The Hipparcos catalogue number (*HIP*). (b) Their *right ascension* ($\alpha - \alpha$) [h m s]. (c) Their *declination* ($\delta - d$) [$^{\circ}$ ' "]. (d) Their *trigonometric parallax* ($p - \pi$) [$'' \times 10^3$]. (e) Their *proper motion in right ascension* multiplied by $\cos d$ ($\mu_{\alpha} \cos d - \mu_{\alpha} \cos d$) [$'' \times 10^3/\text{yr}$]. (f) Their *proper motion in declination* ($\mu_d - \mu_d$) [$'' \times 10^3/\text{yr}$]. (g) Their radial velocity ($v_r - v_r$) [km/s].

HIP	alpha			delta			p	mu_axcosd	mu_d	v_r
13834	2	58	5.08	20	40	7.7	31.41	234.79	-31.64	28.10
14838	3	11	37.67	19	43	36.1	19.44	154.61	-8.39	24.70
18170	3	53	9.96	17	19	37.8	24.14	143.97	-29.93	35.00
18735	4	0	48.69	18	11	38.6	21.99	129.49	-28.27	31.70
19554	4	11	20.20	5	31	22.9	25.89	146.86	5.00	36.60
20205	4	19	47.53	15	37	39.7	21.17	115.29	-23.86	39.28
20261	4	20	36.24	15	5	43.8	21.20	108.79	-20.67	36.20
20400	4	22	3.45	14	4	38.1	21.87	114.04	-21.40	37.80
20455	4	22	56.03	17	32	33.3	21.29	107.75	-28.84	39.65
20542	4	24	5.69	17	26	39.2	22.36	109.99	-33.47	39.20
20635	4	25	22.10	22	17	38.3	21.27	105.49	-44.14	38.60
20711	4	26	18.39	22	48	49.3	21.07	108.66	-45.83	35.60
20713	4	26	20.67	15	37	6.0	20.86	114.66	-33.30	40.80
20842	4	28	0.72	21	37	12.0	20.85	98.82	-40.59	37.50
20885	4	28	34.43	15	57	44.0	20.66	104.76	-15.01	40.17
20889	4	28	36.93	19	10	49.9	21.04	107.23	-36.77	39.37
20894	4	28	39.67	15	52	15.4	21.89	108.66	-26.39	38.90
20901	4	28	50.10	13	2	51.5	20.33	105.17	-15.08	39.90
21029	4	30	33.57	16	11	38.7	22.54	104.98	-25.14	41.00
21036	4	30	37.30	13	43	28.0	21.84	108.06	-19.71	38.80
21039	4	30	38.83	15	41	31.0	22.55	104.17	-24.29	39.56
21137	4	31	51.69	15	51	5.9	22.25	107.59	-32.38	36.00
21152	4	32	4.74	5	24	36.1	23.13	114.15	6.17	39.80
21459	4	36	29.07	23	20	27.5	22.60	109.97	-53.86	43.30
21589	4	38	9.40	12	30	39.1	21.79	101.73	-14.90	44.70
21683	4	39	16.45	15	55	4.9	20.51	82.40	-19.53	35.60
22044	4	44	25.77	11	8	46.2	20.73	98.87	-13.47	39.60
22157	4	46	1.70	11	42	20.2	12.24	67.48	-7.09	43.00
22176	4	46	16.78	18	44	5.5	10.81	73.03	-69.79	44.11
22203	4	46	30.33	15	28	19.6	19.42	91.37	-24.72	42.42
22565	4	51	22.41	18	50	23.8	17.27	79.66	-32.76	36.80
22850	4	54	58.32	19	29	7.6	14.67	63.32	-28.41	38.40
23497	5	3	5.70	21	35	24.2	20.01	68.94	-40.85	38.00
23983	5	9	19.60	9	49	46.6	18.54	63.54	-7.87	44.16
24019	5	9	45.06	28	1	50.2	18.28	55.86	-60.57	44.90



Import the *txt* file in *MS Excel*

- Convert the coordinates in degrees (with 4 decimal points).
- Calculate the angular distance, φ , between each of the stars and the point of convergence, which is at ($\alpha_c = 6^h 7^m$, $\delta_c = +6^\circ 56'$).
- Calculate the proper motion of each star, μ ["/yr], using $\mu_\alpha \cos \delta$ and μ_δ given in the list.
- Use the above data to calculate the distance, r_μ , for each star using the following equation:

$$r_\mu = \frac{v_r \tan \varphi}{4.74047 \mu}$$

where r_μ is the distance of the star in parsecs, v_r is the radial velocity of the star in km/sec, φ is the *angular distance* between the star and the point of convergence that you have already estimated in step 3, while μ is the total proper motion estimated in step 4. Do all stars belong to the Hyades cluster? You can assume that any stars whose distance from the centre of the cluster ($r_\mu = 46.34$ pc) is larger than 10 pc, are not part of the cluster.

- Independently, calculate the distance, r_π , of each star in the list using the trigonometric parallax angle, π .
- Find the average distance of the Hyades cluster, $\overline{r_\mu}$ and $\overline{r_\pi}$, and its standard deviation, σ_μ and σ_π , for each method (*moving cluster* and *trigonometric parallax* methods).
- Which method is more accurate: (i) the *moving cluster* method, (ii) the *trigonometric parallax* method? Please answer with (i) or (ii).

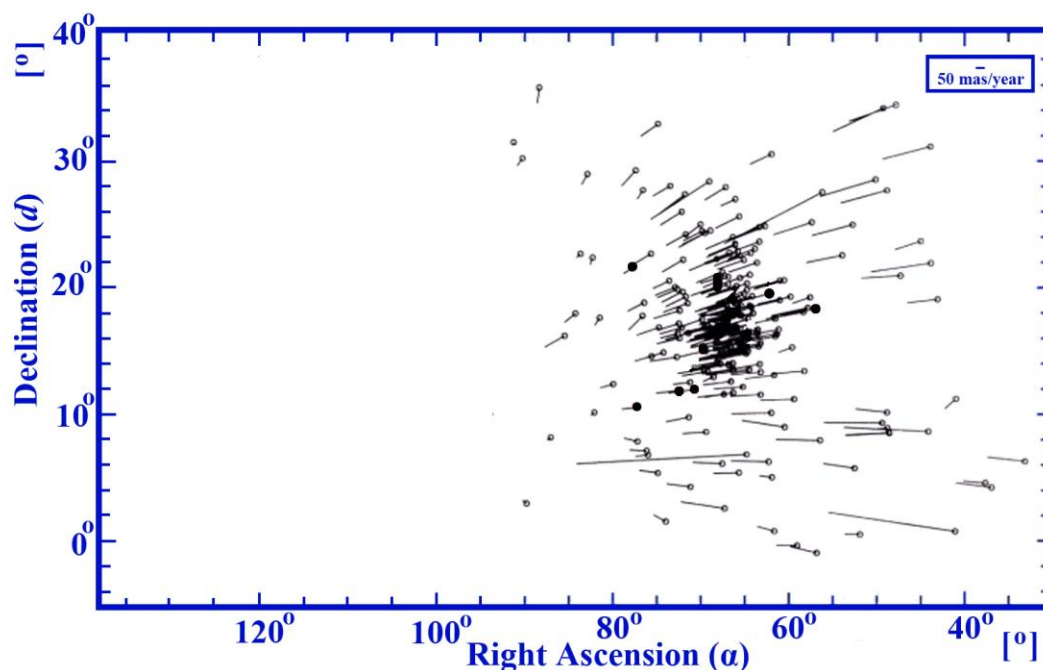


Figure 4. The distance of the Hyades open cluster using the *moving cluster* method

Answer: (1) The student should be able to import the ascii data of the *txt* file in the *MS Excel* spreadsheet application. **(12 Points)**

(2) In order to calculate the angular distance φ (next step), all coordinates should be converted into decimal degrees. The students should be able to convert the [hours, min, sec] and the [°, ', "] into decimal degrees, using . For the right ascension: $\alpha_{\text{deg}} = h \times 15 + \frac{m}{60} + \frac{s}{3600}$ and for the declination:

$\delta_{\text{deg}} = \circ + \frac{'}{60} + \frac{''}{3600}$. All coordinates are positive, so the student does not have to worry about checking the sign, which is not trivial. **(12 Points)**

(3) Using the cosine law, $\varphi = \arccos(\sin \delta_1 \times \sin \delta_2 + \cos \delta_1 \times \cos \delta_2 \times \cos (a_1 - a_2))$, the angular distance, φ , between each of the stars and the point of convergence should be calculated **(15 Points)**

(4) The student should easily calculate the proper motion of each star, by inserting the given equation, $\mu = \sqrt{(\mu_a \cos \delta)^2 + \mu_d^2}$, in the spreadsheet. **(9 Points)**

(5) Again, inserting the given equation, $r_\mu = \frac{v_r \tan \varphi}{4.74047 \mu}$ in the spreadsheet (remembering to divide the given values of μ by 1000 to get arcseconds), the student should be able to calculate the distance, r_μ , of each star. Any star whose distance from the centre of the cluster is larger than 10 pc should be omitted from the following calculations. **(12 Points)**

(6) The trigonometric parallax distance is given by $r_\pi = \frac{1}{\pi \times 10^{-3}}$. This equation should be inserted in the spreadsheet. The result is in parsecs [pc]. **(6 Points)**



7th International Olympiad on Astronomy & Astrophysics
27 July – 5 August 2013, Volos Greece



(7) Using the statistical functions *AVERAGE* and *STDEV* of MS Excel, the student should easily calculate the *average* and the *standard deviation* of r_μ and r_π . **(6 Points)**

(8) The method with the smaller standard deviation is, obviously more accurate. **(3 Point)**

(*macros are allowed*)

GROUP COMPETITION



7th International Olympiad
on Astronomy & Astrophysics
VOLOS 2013 HELLAS

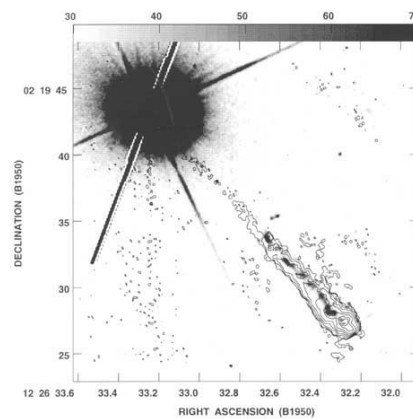
A.

i. first astronomer to suggest that the Earth is not the center of the Universe? [1] – 2

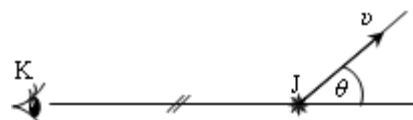
1. Galileo
2. Aristarchus
3. Copernicus
4. Cassini
5. Zhang Heng
6. al-Biruni
7. Kepler
8. Brahe
9. Aryabhata

ii. The obliquity of the ecliptic in degrees. (one decimal) [3] – 234

iii. The first known and brightest quasar is 3C_.
[3] – 273



iv. A relativistic jet moves at $0.83c$. If no Doppler shift is observed, how many degrees is the angle between the jet and the line of sight, assuming that the source has negligible velocity? (integer)
[3] – 122

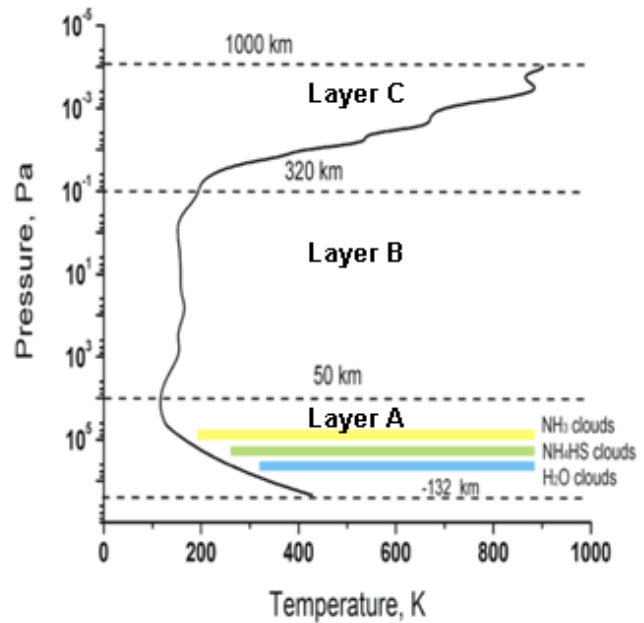


v. This nebula is NGC_. [4] - 7000



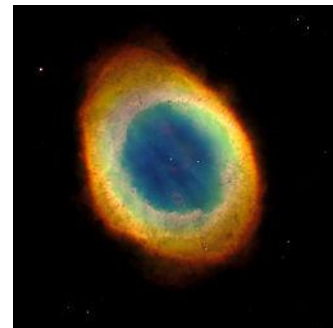
- vi. Choose the names of the layers A, B and C of the Jovian atmosphere, beginning from Layer A. [3] – 195

1. Troposphere
2. Magnetosphere
3. Ionosphere
4. Ozone layer
5. Thermosphere
6. Lithosphere
7. Chromosphere
8. Photosphere
9. Stratosphere



B.

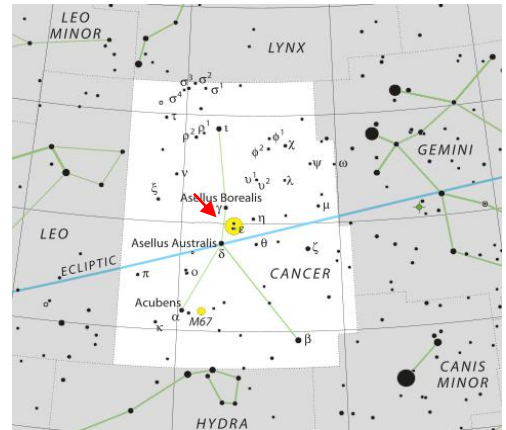
- i. How many AU is one parsec? (integer) [6] – 206265
- ii. Jupiter–Sun–Trojan Asteroids angle (degrees). (integer) [2] – 60
- iii. This nebula is M_. [2] – 57



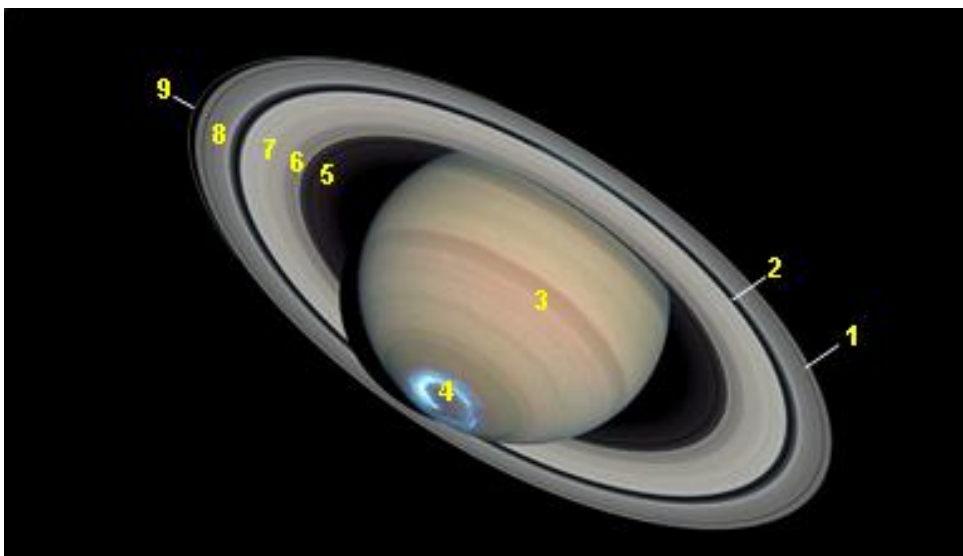
- iv. On the 25th of August, at a latitude of $\phi > 0$ and longitude $\lambda_L = 37^\circ W$ (time zone = GMT - 2hrs), we observe a star with Declination (δ) such that $\phi + \delta = 90$, and Right Ascension $\alpha = 67.5^\circ$. What is the local civil time at the hh:mm during the lower culmination of the star? It is given that at 00:00hrs on the 24th of August, hour angle of vernal equinox is $21^h 58^m$. [4] – 1857
- v. During a Meteor Shower, in a radius of 100km, an observer counted 600 meteors/min. If the rest frame velocity of the meteors was 10km/s with opposite direction to that of the Earth's orbital velocity, what is the mean distance between two meteors in km? Assume cylindrical geometry. (integer) [2] – 50
- vi. The photon's spin. [1] – 1

C.

- i. Difference between sidereal time of two places is $2^h 47^m 24^s$. The difference in their longitude is (in deg and min). [4] – 4151
- ii. A galaxy in the constellation Triangulum is M_. [2] – 33
- iii. Pluto's moons. [1] – 5
- iv. The Beehive Cluster is M_. [2] – 44

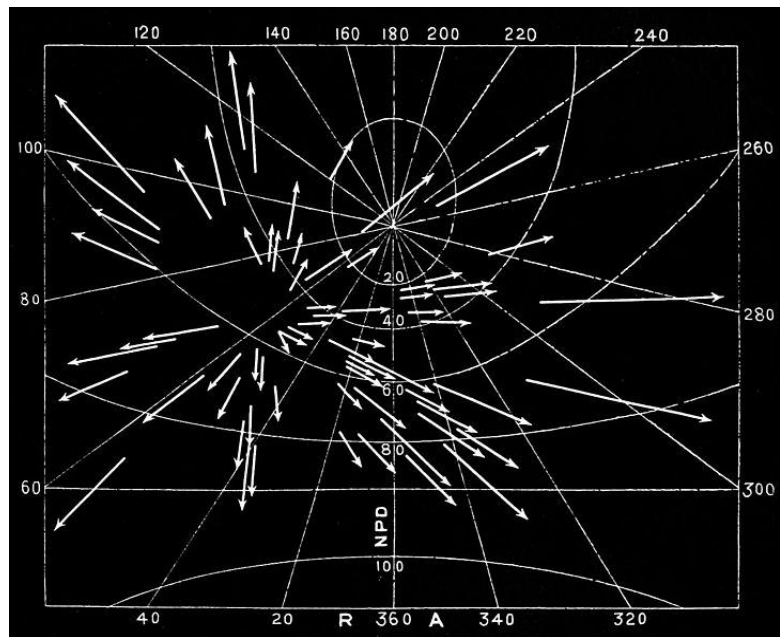


- v. Tidal forces are proportional to R^{-n} . Value of n is [1] – 3
- vi. Proxima Centauri is $4.243ly$ distant. It has apparent magnitude of $11.5mag$ and approaches Earth at $21.7km/s$. After how many thousands of years will it be visible to the naked eye? (integer) [2] – 54
- vii. Cassini Division, D Ring, Encke Gap, A Ring [4] – 2518



viii. The radiant of this meteor shower is in: [1] – 4

- 1.Cygnus
- 2.Gemini
- 3.Leo
- 4.Perseus
- 5.Cetus
- 6.Orion



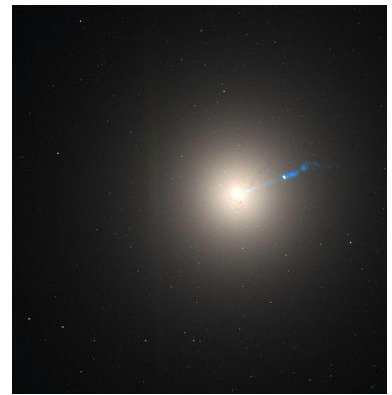
D.

i. At this Lagrange point on the orbit of Jupiter lies a Trojan camp! [1] – 5

ii. Kiloparsec to light-years. (integer) [2] – 3262

iii. The mean distance of Venus from Sun (AU). (two decimals) [3] – 072

iv. The supergiant elliptical galaxy near the centre of the Virgo Galaxy Cluster is M_. [2] – 87



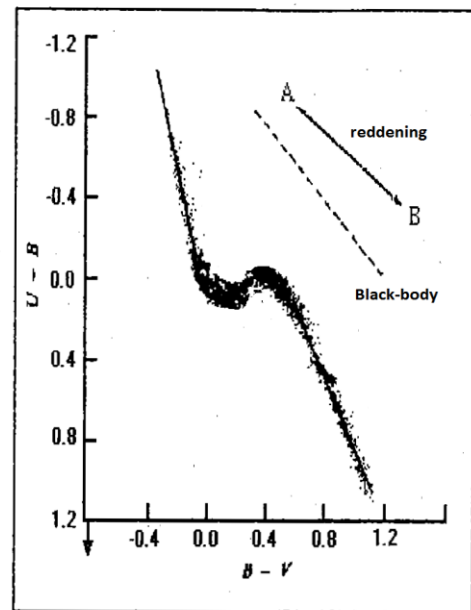
v. The molar mass of the substance that forms clouds in the upper atmosphere of Venus. [2] – 98

vi. spin of an electron (one decimal) [2] – 05

vii. Capella's parallax is 77.3 milliarcseconds. Find its distance in light years (2nd decimal) [3] – 422

E.

- i. A A_0V star has observed Colour Index $B-V=0.70$. How many times brighter would the star look if there was no interstellar extinction? Consider that $A_V = 3 \times E_{B-V}$, where E_{B-V} is the Colour Excess and A_V is the Visual Absorption. (integer) [1] – 7
- ii. The sidereal rotation period at the equator of the Sun (days). At the spectrum of the Sun, the line H_α ($\lambda=6,563\text{\AA}$) has a broadening of $0.089\text{\AA}$. (integer) [2] – 25
- iii. One star has a B-V colour index of 0.2mag and a U-B colour index of -0.1mag . What is its Colour Excess E_{B-V} ? (one decimal) [2] – 03

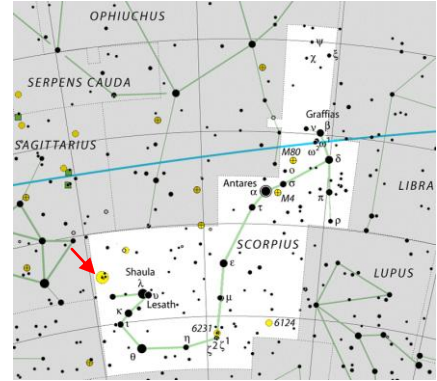


- iv. Chandrasekhar Mass limit in solar radii. (two decimals) [3] – 144
- v. The orbital period of Mercury (days). (integer) [2] – 88
- vi. The axial tilt of an exoplanet without atmosphere is 0° . Its orbital period is really long. How many times greater is the mean temperature of the illuminated (by the corresponding star) hemisphere when the planet spins really slowly compared to when it spins really quickly? (one decimal) [2] – 12
- vii. The atomic mass (in a.m.u.) of the heaviest element which can be produced in a star, before it meets its end. [2] – 56
- viii. Diameter of the biggest single aperture optical telescope in meters (integer) [2] – 10
- ix. The inventor of the telescope. [1] – 4
 - 1.Demisianos
 - 2.Brache
 - 3.Ibn Sahl
 - 4.Lippershey
 - 5.Huygens
 - 6.Harriot
 - 7.Marius
 - 8.Newton
 - 9.Galileo

F

- i. A binary system is consisted of two stars that revolve around the centre of mass with semi-major axes of 11.25 A.U. And 18.62 A.U. If the total mass of the system is 5 solar masses, calculate the period (in years) of the system. (integer) [2] – 73

- ii. Ptolemy's Cluster is M_. [1] – 7



- iii. What will be the diameter of the telescope (in meters) required to resolve a planet revolving around a star with orbital radius 1 A.U., if the distance to star is 226.26 light years? Ignore seeing. (one decimal) [2] – 96

- iv. Castor is a multiple star of how many components? [1] – 6

- v. The picture shows SNR __. [4] – 1054



- vi. Luminosity class of sub-dwarfs [1] – 6

- vii. Alcyone, Taygeta, Maia, Electra, Atlas [5] – 51836



- viii. Luminosity class of the Sun [1] – 5

G

i. Bode's Galaxy is $M_{\odot}$. [2] – 81

ii. Cigar Galaxy is $M_{\odot}$. [2] – 82

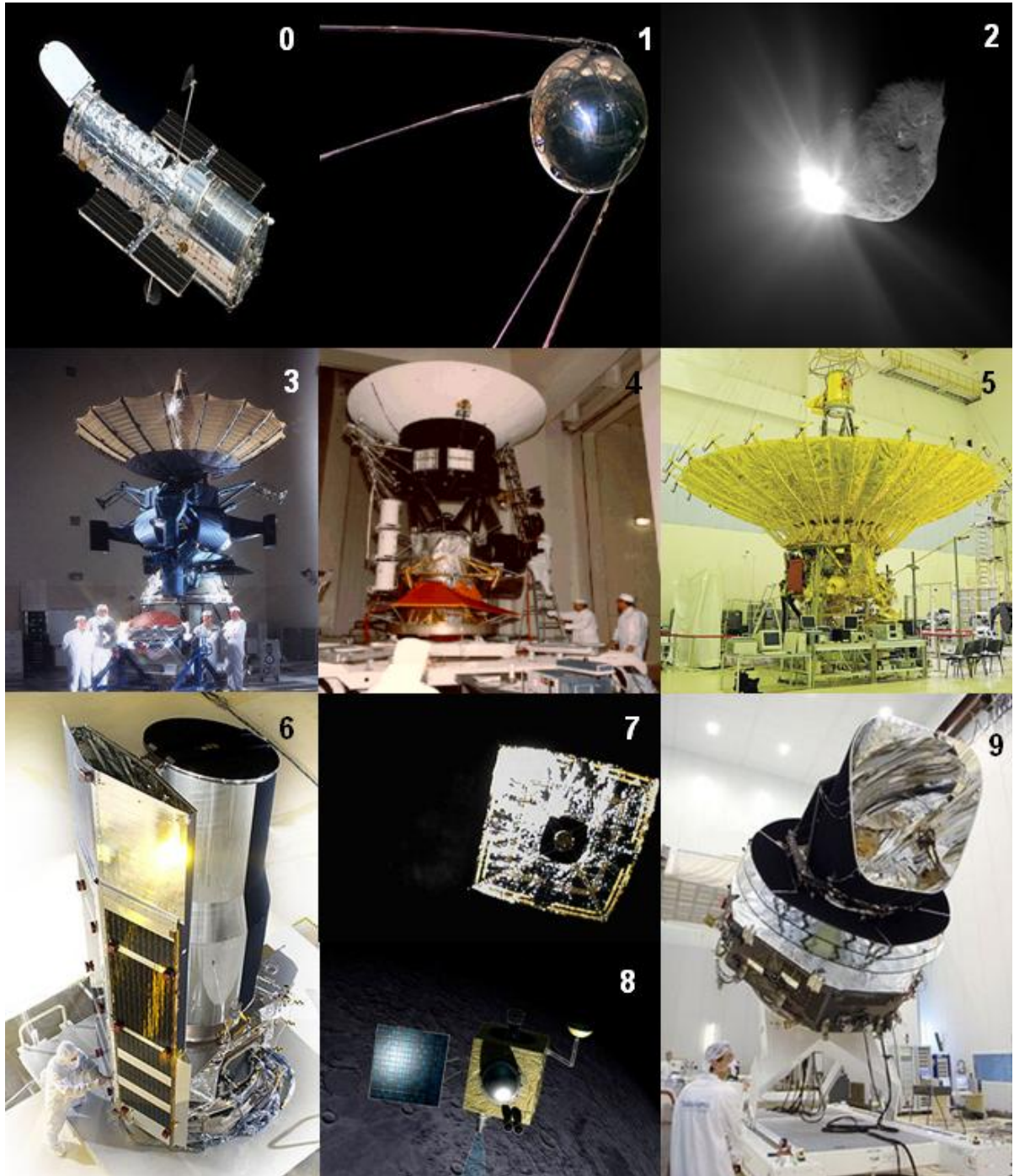


iii. How many times fainter will a star become if its magnitude is increased by 1mag? (three decimals) [4] – 2512

iv. A Globular Cluster near the nostril of Pegasus is $M_{\odot}$. [2] – 15

v. Hipparchus described a Solar Eclipse which was seen as Total in the Hellespont ($40^{\circ} 25'N$, $29^{\circ} 43'E$) but Partial in Alexandria ($31^{\circ} 12'N$, $29^{\circ} 55'E$), where at maximum $4/5$ of the Sun was hidden. Hipparchus calculated the parallax of the limb of the Moon between the Hellespont and Alexandria, assuming that the parallax of the Sun is 0. He also knew that the Moon has the same apparent angular diameter as the Sun, with a value of $1/650$ of the circle. Finally, by applying simple trigonometry, he calculated the approximate distance between the Earth and the Moon in Earth radii. How many Earth radii is the Earth–Moon distance? (integer) [2] – 64

vi. Chandrayan-1, Deep Impact, Planck Telescope, Hubble Space Telescope [4] – 8190



vii. If the mass of a star is 5 solar masses , and its initial radius and rotation period is $21 \times 10^5 \text{ km}$ and 4.083 days respectively, how many microseconds will be its period when it turns into a Pulsar with a radius of 10 km ? Assume the star is a homogenous sphere and does not lose mass. [1] – 8

H.

- i. How many planets of the Solar System have rings? [1] – 4
- ii. Angle corresponding to $16^{\text{h}} 16^{\text{m}}$ (integer) [3] – 244
- iii. The Saros in _y _d. [4] – 1811
- iv. If a more accurate indicator existed for the latitude scale of the telescopes you are going to use during the observational part, it would show _° _'. [4] – 3922
- v. R.A. of Duschbba i.e. δ Scorpil (hh). [2] – 16
- vi. An asteroid is $2.84U$ away from the Sun. At opposition, its visual brightness was $1.46 \times 10^{-16} \text{ Watt/m}^2$. What is its radius (in meters) if its albedo is 1.00? (integer) [3] – 496

I

i. On 11th December 2117 a Venus Transit will take place. How many years later is the next one? [1] – 8

ii. An Open Cluster in Canis Major is M_. [2] – 41

iii. Write the numbers of the three brightest stars in descending order of brightness: [3] – 324

1. Betelgeuse
2. Vega
3. Canopus
4. α Centauri
5. Capella
6. Procyon
7. Altair
8. Rigel
9. Arcturus

iv. A circumpolar star's altitude at upper culmination is 76.8° and at lower culmination is 10° . What is the latitude of the place? [3] – 434

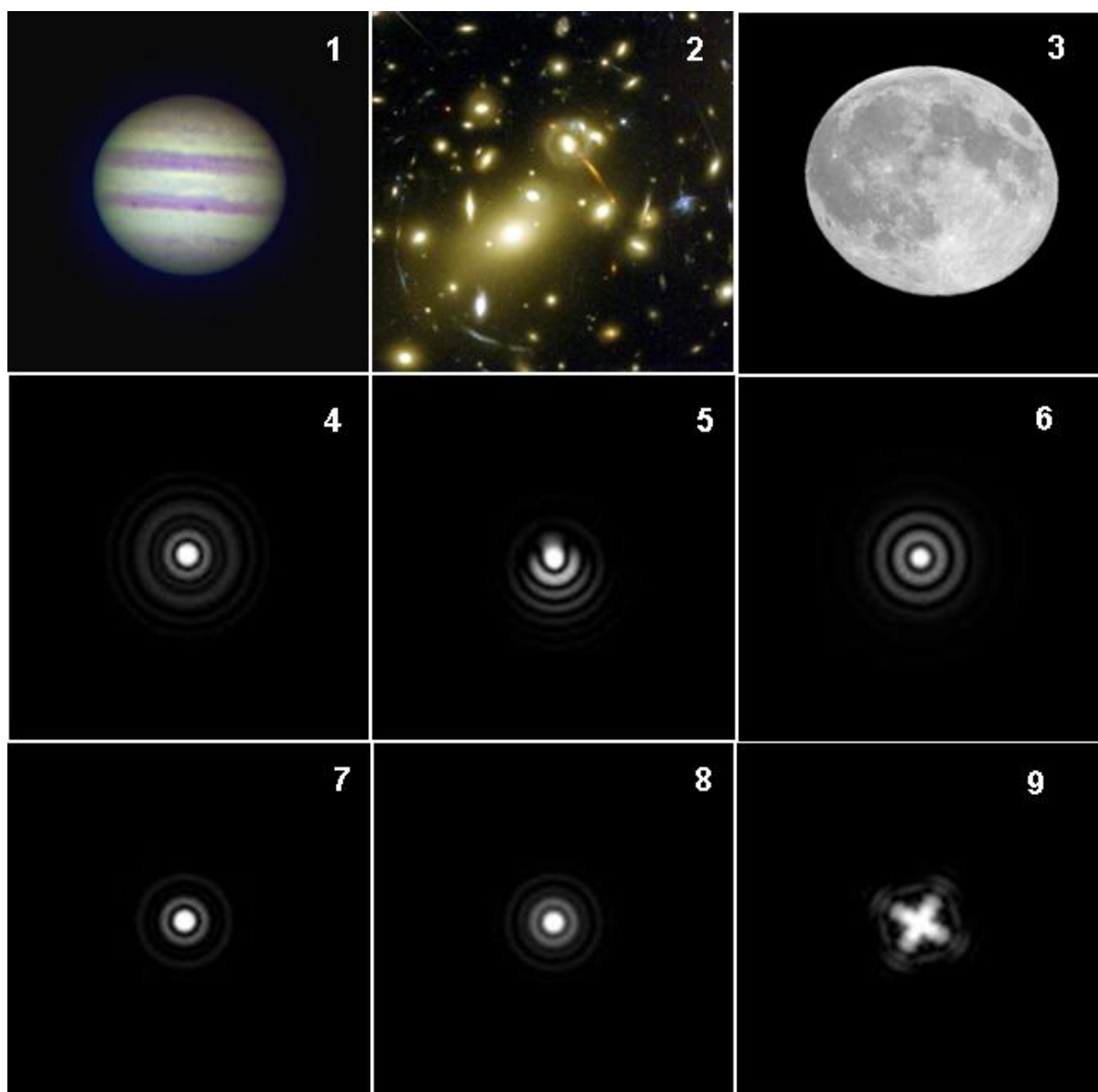
v. Leo is Nth zodiacal sign. N is [1] – 5

vi. The Solar System lies within this arm of the Milky Way: [1] – 4

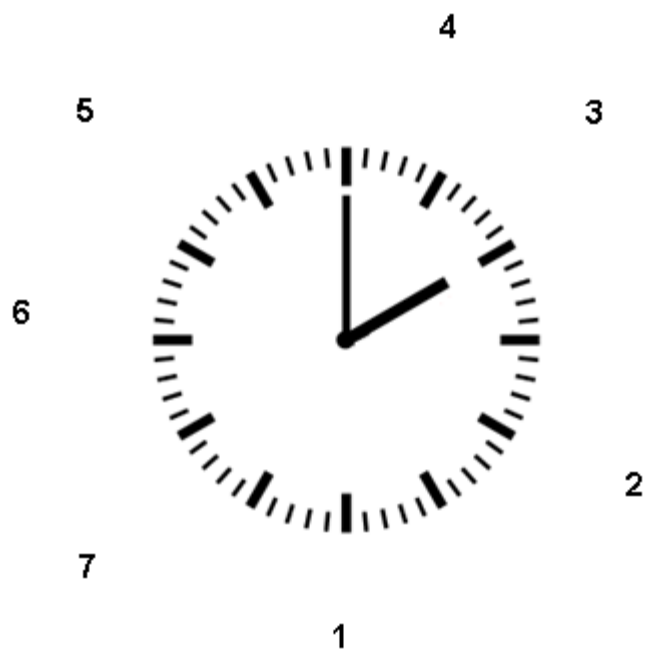
1. Sagittarius–Carina
2. Scutum–Crux
3. Norma
4. Orion
5. Perseus
6. Cygnus
7. Outer

vii. The resonance between Pluto and Neptune is _:_ . [2] – 23

viii. Distortion, Coma, Astigmatism [3] – 359

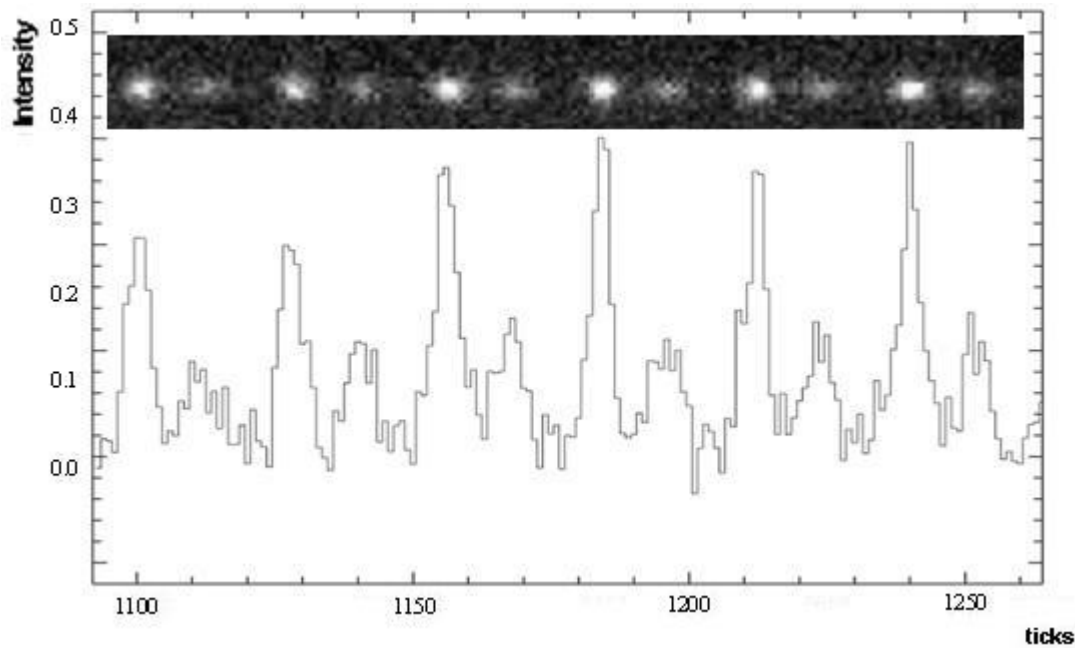


ix. On 16th December you visit a country of the Southern Hemisphere which uses Daylight Saving Time. When it is 2 o' clock, you place your watch so as the hour hand points to the projection of the Sun to the horizon. Which number indicates North (approximately)? [1] – 3



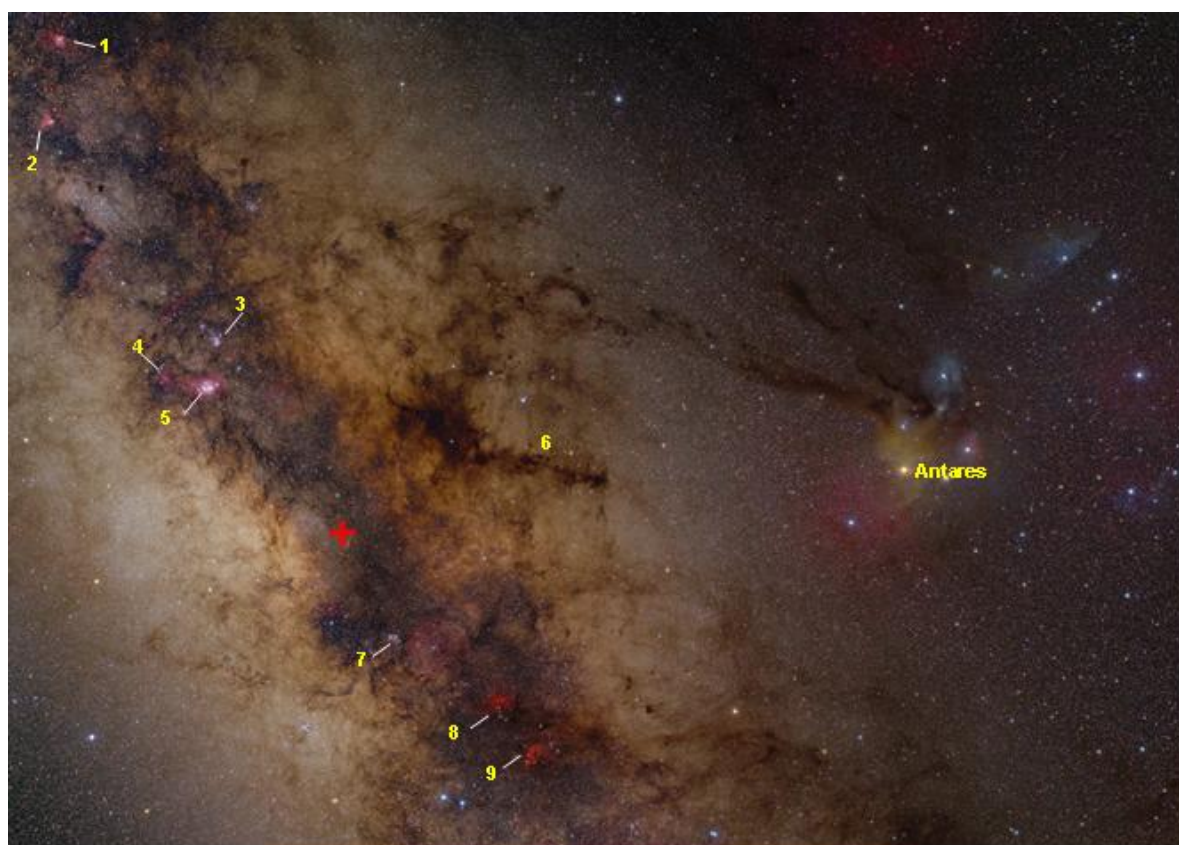
J.

- i. Each Zodiacal Sign extends for that many degrees along the ecliptic. [2] – 30
- ii. You are given the light curve of a pulsar. What is its rotation period (ms) if the time interval between two “ticks” is 1.1965ms? (one decimal) [3] – 335



iii.
Pipe
Nebu
la,
Trifi
d
Nebu
la,
Lago
on
Nebu
la,
Cat'
s

Paw Nebula, Eagle Nebula [5] – 63591



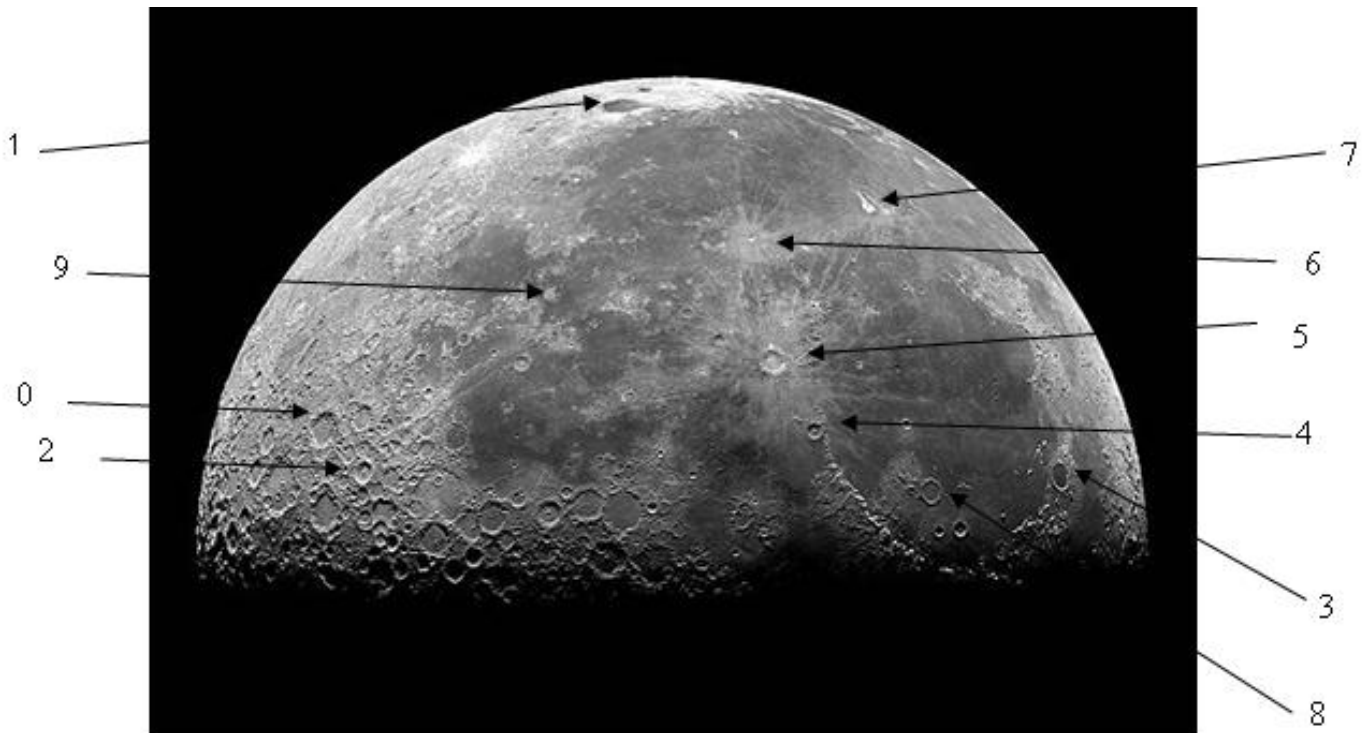
- iv. Write the corresponding numbers of the three biggest asteroids in descending order: **[3]**
– **248**

1. Hygiea
2. Ceres
3. Eugenia
4. Pallas
5. Fortuna
6. Aurora
7. Nemesis
8. Vesta
9. Psyche

- v. Baade and Hubble measured the distance to NGC 1049 to be 188 kpc. Thus the distance modulus ($m - M$) they would get for this galaxy is (2 decimal points) **[4]** – **2137**

K.

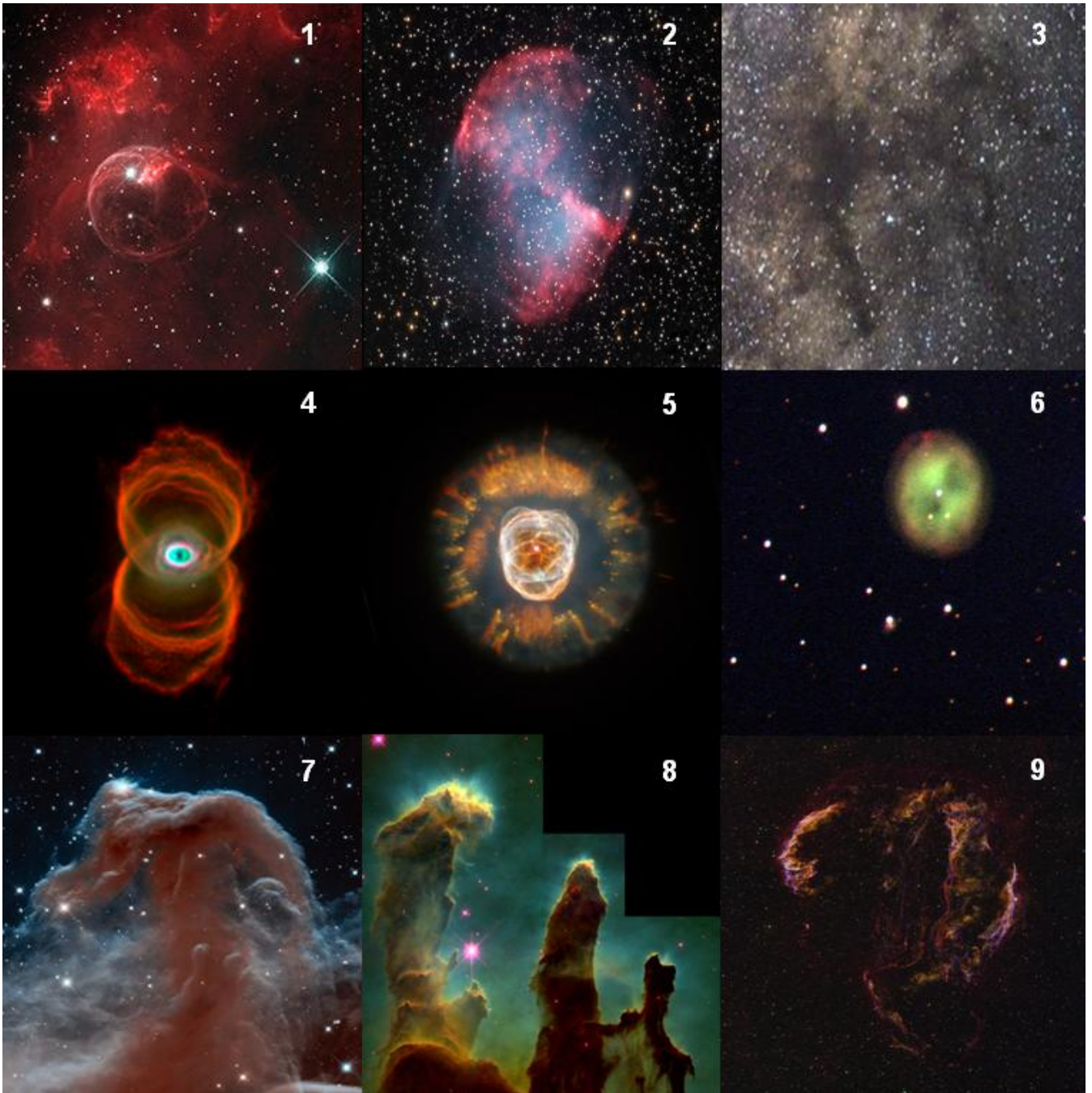
- i. Two stars have angular separation of 13.84 arcsec. We are photographing them with a f/10 telescope with a diameter of 20 cm. What is their linear separation on the photograph in micrometers? (integer) [4] - 1342
- ii. Grimaldi, Plato, Tycho, Eratosthenes [4] - 1324



- iii. How many degrees under the horizon is the centre of the Sun when the Astronomical Twilight ends? [2] - 18
- iv. The mission of Apollo ___ was aborted after an oxygen tank exploded on the way to the Moon. [2] - 13
- v. If the Observatory Factor is 0.95, what is the Wolf Number if the number of sunspot groups is 13 and the number of individual spots is 14? (integer) [3] - 137
- vi. A star is 100pc away. Its Apparent Visual Magnitude is $m_v=13.0mag$, its Apparent Photographic Magnitude is $m_{pg}=14.6mag$ and its Absolute Visual Magnitude is $M_v=4.7mag$. What is its Colour Excess? Consider that $A_v = 3 \times E_{B-V}$, where E_{B-V} is the Colour Excess and A_v is the Visual Absorption. (one decimal) [2] - 11

L.

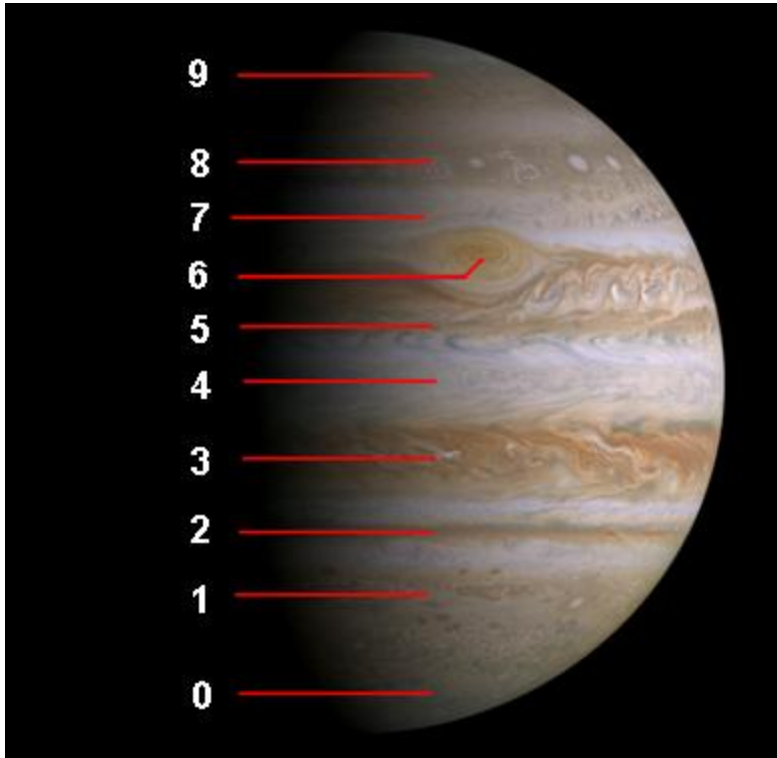
- i. Pillars of Creation, Dumbbell Nebula, Dark Horse Nebula (Great Dark Horse), Bubble Nebula, Engraved Hourglass Nebula [5] – 82314



- ii. What is the wavelength of the spectral line $H \beta$? (integer) [4] – 4862
- iii. First planet to be discovered using Telescope (Mercury is 1, Neptune is 8) [1] – 7
- iv. Consider an eclipsing binary, with central eclipses. Time between the first and fourth contact of primary eclipse is 1.5 hours and time between the 2nd and 3rd contact is 1 hour. Find the ratio of their radii. (integer) [1] – 5
- v. The final Apollo lunar mission is Apollo __. [2] – 17
- vi. How many times greater is the escape velocity of a satellite compared to the velocity that is required for an orbit with radius equal to the celestial body's radius? (one decimal) [2] – 14
- vii. A star is behind the Coalsack Nebula in a distance of $200pc$. If its apparent magnitude is $m=18mag$ and the optical depth of the Nebula is $\tau=1.38$, what is its Absolute Magnitude? (integer) [2] – 10

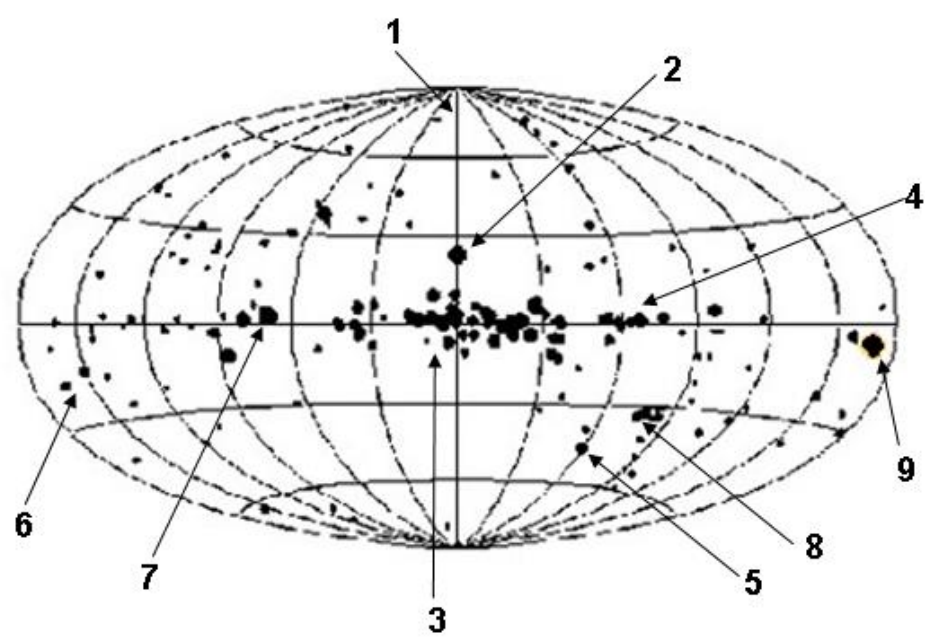
M.

- i. Great Red Spot, North Polar Region, South Equatorial Belt, Equatorial Zone [4] – 6054



- ii. This is a map of the sky in X-rays using galactic coordinates. [5] – 82971

- ÷ Large Magellanic Cloud
- ÷ Scorpio X-1
- ÷ Cancer Nebula
- ÷ Cygnus X-1
- ÷ Coma



- iii. The table shows the composition of the atmosphere of Mars, Earth, Jupiter, Neptune and Venus. Find Jupiter, Venus, Neptune. [3] – 532

	1	2	3	4	5
H ₂	0.000055%	80.0	-	-	86.4
He	0.000524%	19.0	12 ppm	-	13.6
CH ₄	0.000179%	2	-	10.5 ppb	0.0018
NH ₃	trace	<600 ppb	-	-	0.0006
H ₂ O	locally 0.001%–5%	-	20 ppm	210 ppm	520 ppm
H ₂ S	-	<3 ppm	-	-	70 ppm
CO ₂	0.0397%	0.3 ppb	96.5%	95.32%	30 ppb
HCN	-	60 ppb	-	-	60 ppb
N ₂	78.084%	-	3.5%	2.7%	-
O ₂	20.946%	-	-	1300 ppm	-
Ar	0.9340%	-	70 ppm	1.6%	-
H ₂ SO ₄ *	-	-	150 ppm	-	-

- iv. Lagoon nebula is M __. [1] – 8

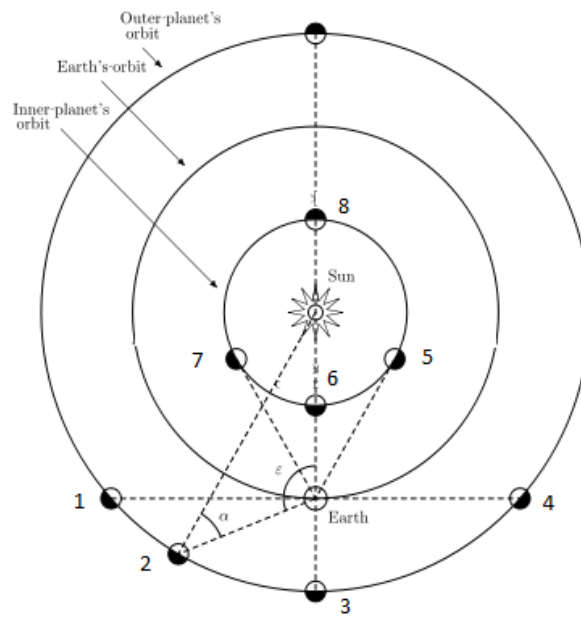


- v. Sequence in which following discoveries received Nobel prizes. [4] – 4253

1. Cepheid Period–Luminosity relation
2. Discovery of pulsars
3. Accelerated Expansion of the Universe
4. Discovery of cosmic rays
5. Chandrasekhar Mass Limit

A.

- i. M31 is NGC_. [3] – 224
- ii. The effective temperature of the solar surface in Kelvin. (integer) [4] – 5778
- iii. Sun's Absolute Visual Magnitude. (two decimals) [3] – 483
- iv. Eastern quadrature, Superior conjunction [2] – 18



- v. Hour angle of setting sun on the equinox day [1] – 6

B.

- i. The Trifid Nebula is M_. [2] – 20

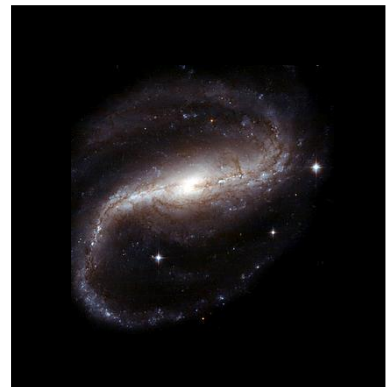


- ii. The temperature at an exoplanet's atmosphere is $336K$. The mean speed of the nitrogen molecules ($m=4.7 \times 10^{-26}kg$) in the same temperature is $0.5km/s$. The mean speed (km/s) that the nitrogen molecules acquire when the temperature becomes four times greater is_. [1] – 1

- iii. The emission spectral line $H\alpha$ is the result of the transition of an electron between energy levels _ to _. [2] – 32

- iv. The morphological class of this galaxy. [1] – 3

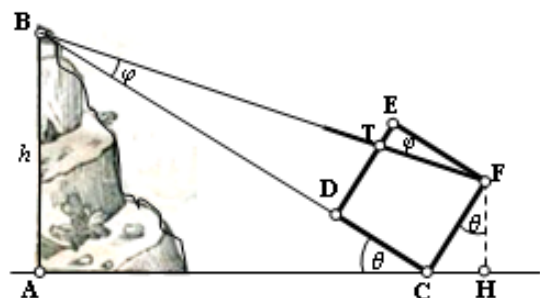
1. SBa
2. SBb
3. SBc
4. S0
5. SBO
6. Irr
7. Sa
8. Sb
9. Sc



- v. Io, Europa and Ganymede revolve in a _:_:_ resonance. [3] – 124

- vi. Vega's B-V Colour Index. [1] – 0

- vii. Find the height of the mountain (metres). $FE/TE=800$ and $CH=0.4m$. EFCD is a square and FH is vertical to CH. [3] – 320



C.

- i. The Julian century has __days. [5] – 36525
- ii. The first to measure Earth's radius, only using data from a single location. [1] – 7

- 1. Posidonius
- 2. Picard
- 3. al-Ma'mun
- 4. Eratosthenes
- 5. al-Farghani
- 6. Snell
- 7. al-Biruni
- 8. Cassini
- 9. Gauss

- iii. This image shows a part of the Virgo Galaxy Cluster. The long chain of galaxies is known as Markarian's Chain. At the bottom right, pointer shows a giant elliptical galaxy (also a powerful radio source) known as M__. [2] – 84



- iv. A star cluster which lies between η and ζ Hercules is M__. [2] – 13



D.

- i. The Orion Nebula is M_. [2] – 42



- ii. This year Galileo observed the sky with a telescope for the first time. [4] – 1609

iii. R

iv. N

v. C



E.

i. What would have been temperature of CMBR at $z=8.63$? (upto 1st decimal) [3] – 263

ii. The axial tilt of the Earth (integer) [2] – 23

iii. Number of planets of the Solar System that have moons. [1] – 6

iv. The wavelength used to chart the Milky Way (in cm). [2] – 21

v. Pleiades is M_. [2] – 45

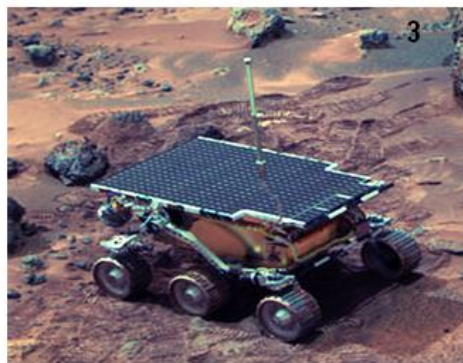
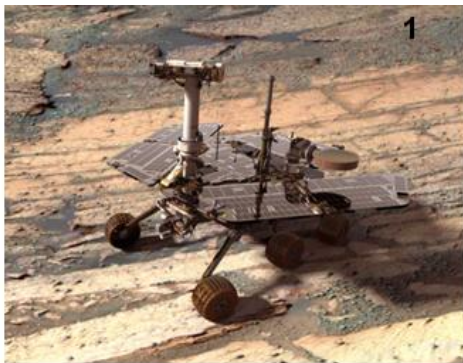
vi. Write scientist number in correct sequence [3] – 148

- He confirmed the Big Bang Theory by observing the CMBR
- He coined the name “Big Bang Theory”
- He introduced Big Bang Nucleosynthesis

1. Robert Wilson
2. Alexander Friedmann
3. Edwin Hubble
4. Fred Hoyle
5. George Lemaitre
6. Fritz Zwicky
7. Arthur Walker
8. George Gamow

F.

- i. Every _ years (integer), Halley's comet appears! [2] – 75
- l.
- ii. The relative error of the spectroscopic parallax is 15% and the absolute error of the trigonometric parallax is $0.005''$. Over which distance (pc) the spectroscopic parallax becomes more accurate than the trigonometric? (integer) [2] – 30
- iii. The molecular weight (in a.m.u.) of the main constituent of Titan's oceans [2] – 16
- iv. 189.08 light years in parsec (integer) [2] – 58
- v. In the beginning of July this planet stopped its retrograde motion. How many AU is its mean orbital radius? (one decimal) [2] – 96
- vi. Sojourner, Lunokhold 1, Curiosity [3] – 342

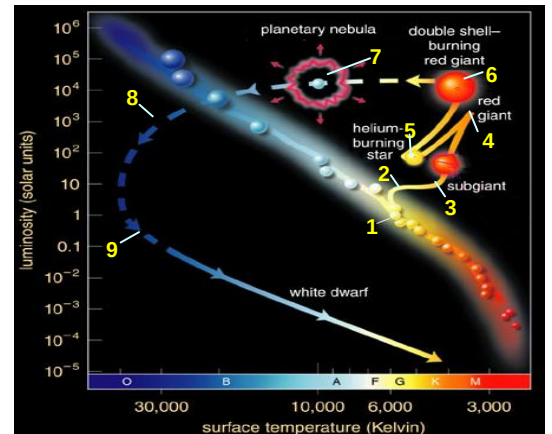


G.

i. The four greatest celestial objects in descending order of size: [4] – 3657

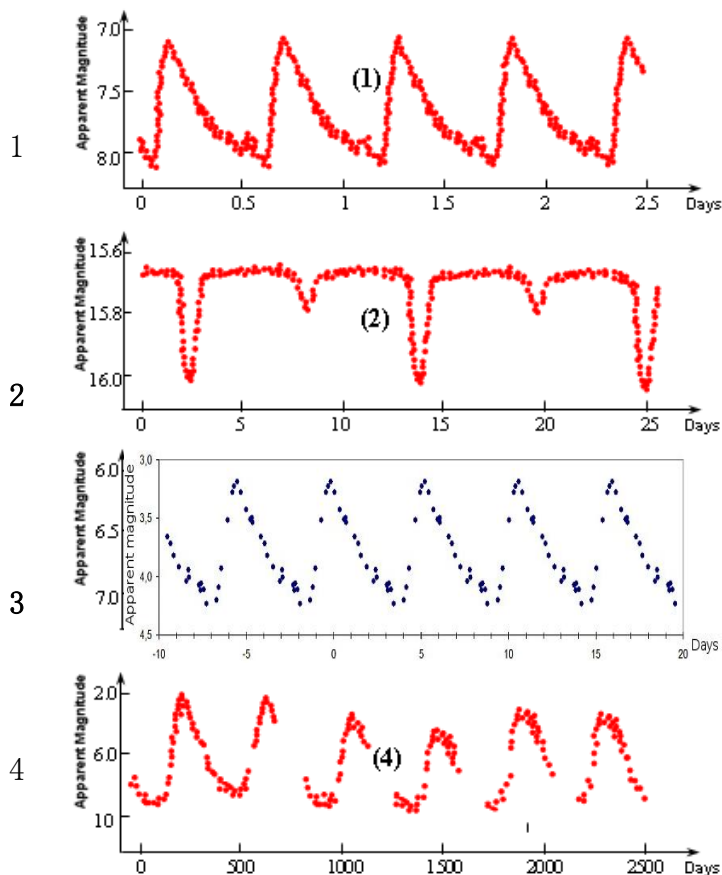
- 1.Moon
- 2.Europa
- 3.Ganymede
- 4.Io
- 5.Mercury
- 6.Titan
- 7.Callisto
- 8.Triton
- 9.Titania

ii. At which point does the Helium Flash occur? [1] – 4



iii. The polarity of the solar magnetic field is reversed every __years. (integer) [2] – 11

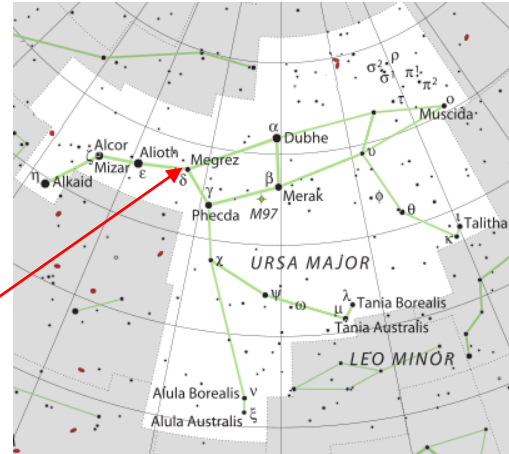
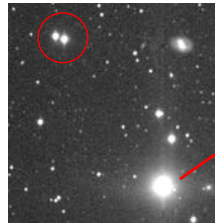
iv.RR Lyrae, Mira, Classical Cepheid, Eclipsing Binary [4] – 1432



v. Days between the longest day in southern hemisphere and the next equinox. [2] – 89

H.

- i. Sombrero galaxy is $M_{\odot}$. [3] – 104
- ii. Number of martian satellites. [1] – 2
- iii. The circle marks is $M_{\odot}$. [2]
– 40



iv. Arrange the following in the increasing order of masses [6] – 213546

- 1. Hyades Cluster
- 2. Eta Carinae
- 3. Omega Centauri
- 4. M31
- 5. LMC
- 6. Virgo Cluster

v. Number of crew members who perished in challenger space shuttle disaster. [1] – 7

I.

- i. Assuming you are seeing the moon in the sky from Greece, how many days have passed since the last New Moon? (integer). [2] – 25

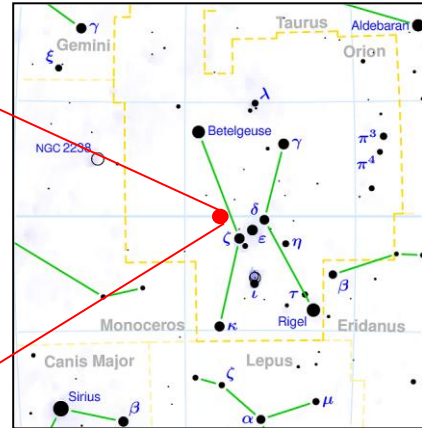


- ii. Difference between the solar day and the sidereal day in minutes. [1] – 4
- iii. The number of constellations [2] – 88
- iv. The Whirlpool Galaxy is M_. [2] – 51
- v. The four closest galaxies to the Milky Way in ascending order: [4] – 3491
- 1.SMC
 - 2.Andromeda
 - 3.Canis Major Irregular Dwarf
 - 4.Sagittarius dwarf irregular galaxy
 - 5.Triangulum galaxy
 - 6.Fornax Dwarf Spheroidal
 - 7.Barnard's galaxy
 - 8.Maffei I
 - 9.LMC
- vi. Number of Van Allen Radiation Belts. [1] – 2
- vii. The L point through which matter from one star of a binary system escapes to the other. [1] – 1

J.

i. The CMBR has a thermal [black body](#) spectrum at a temperature of _Kelvin. (two decimals) [3] – 273

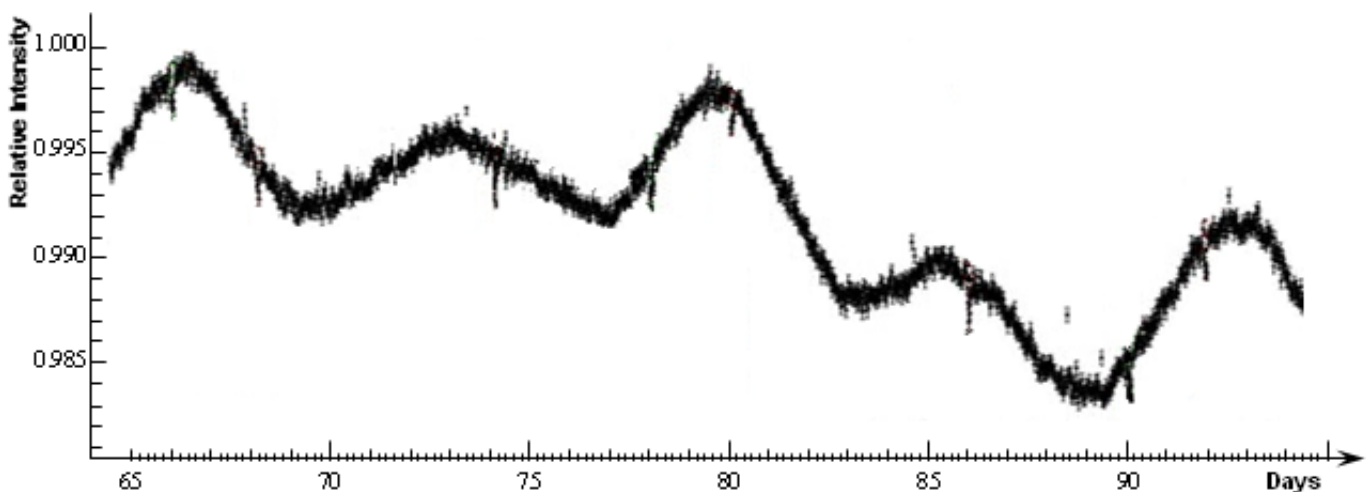
ii. This nebula is M_. [2] – 78



iii. Radiation with an energy of 2.5eV. [1] – 4

1. gamma-ray
2. X-ray
3. Ultraviolet
4. Visible
5. Infrared
6. Far Infrared
7. Microwave
8. Radio
9. Super Low Frequency

iv. Two exoplanets with a radius five times the radius of Earth rotate around a star (radius equal to this of the Sun). Find the shortest orbital period (days). (two decimals) [3] – 595

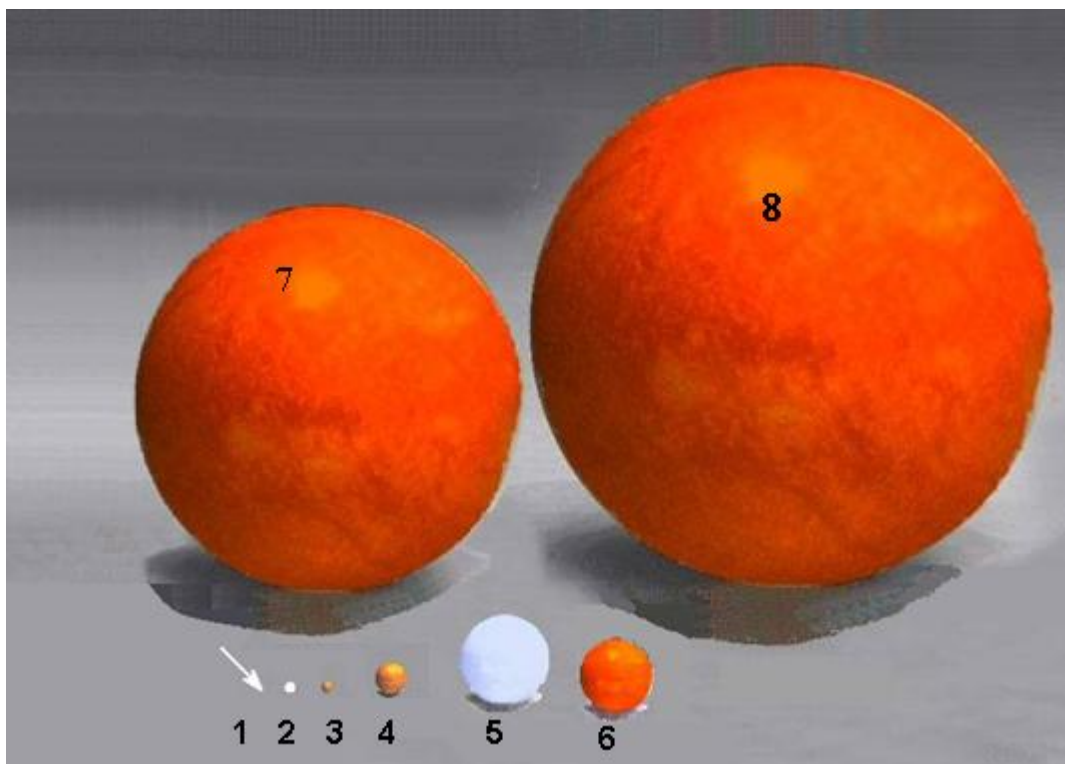


v. The first to calculate the AU by measuring the parallax of a planet: [1] – 1

- 1.Cassini
- 2.Aristachus
- 3.Hipparchus
- 4.Horrocks
- 5.Halley
- 6.Bayly
- 7.Euler
- 8.Lomonosov
- 9.Newcomb

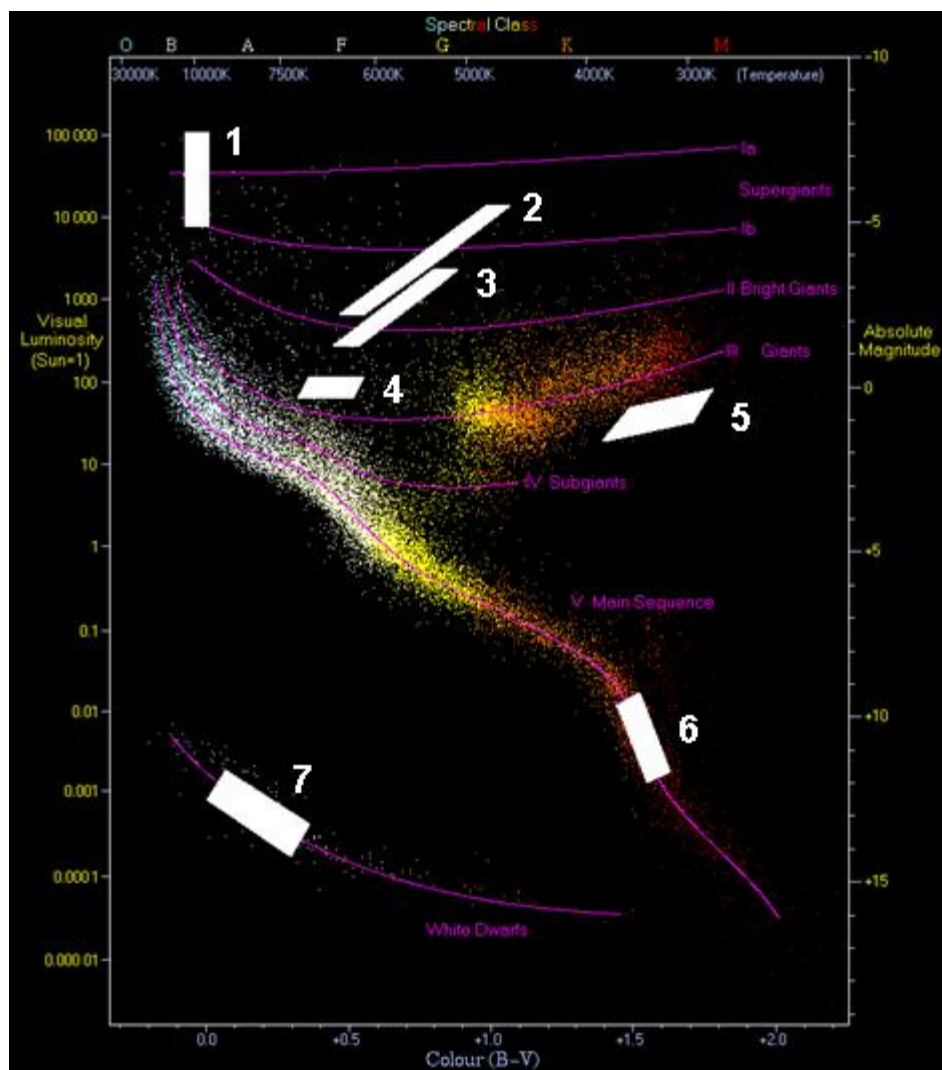
vi. The diameters of the stars (Sirius, Betelgeuse, Aldebaran, Antares, Rigel, Pollux, Arcturus, Sun) are to scale. [3] – 875

Betelgeuse, Antares, Rigel



K.

- i. Every that many years, the orientation of Earth's axial tilt shifts by 1° . (two decimals) [4] – 7159
- ii. When we swim in the sea, the height of our eyes from its level is 20cm . How many kilometres away can we look on the surface of the Earth? (one decimal) [2] – 16
- iii. Flare Stars, Type I Cepheids, RR Lyrae Variables [3] – 624



- iv. The first Pulsar discovered was nicknamed LGM-1. Its official name is now PSR J1921 +2153. Its declination is $_\circ_\prime$. [4] – 2153

L.

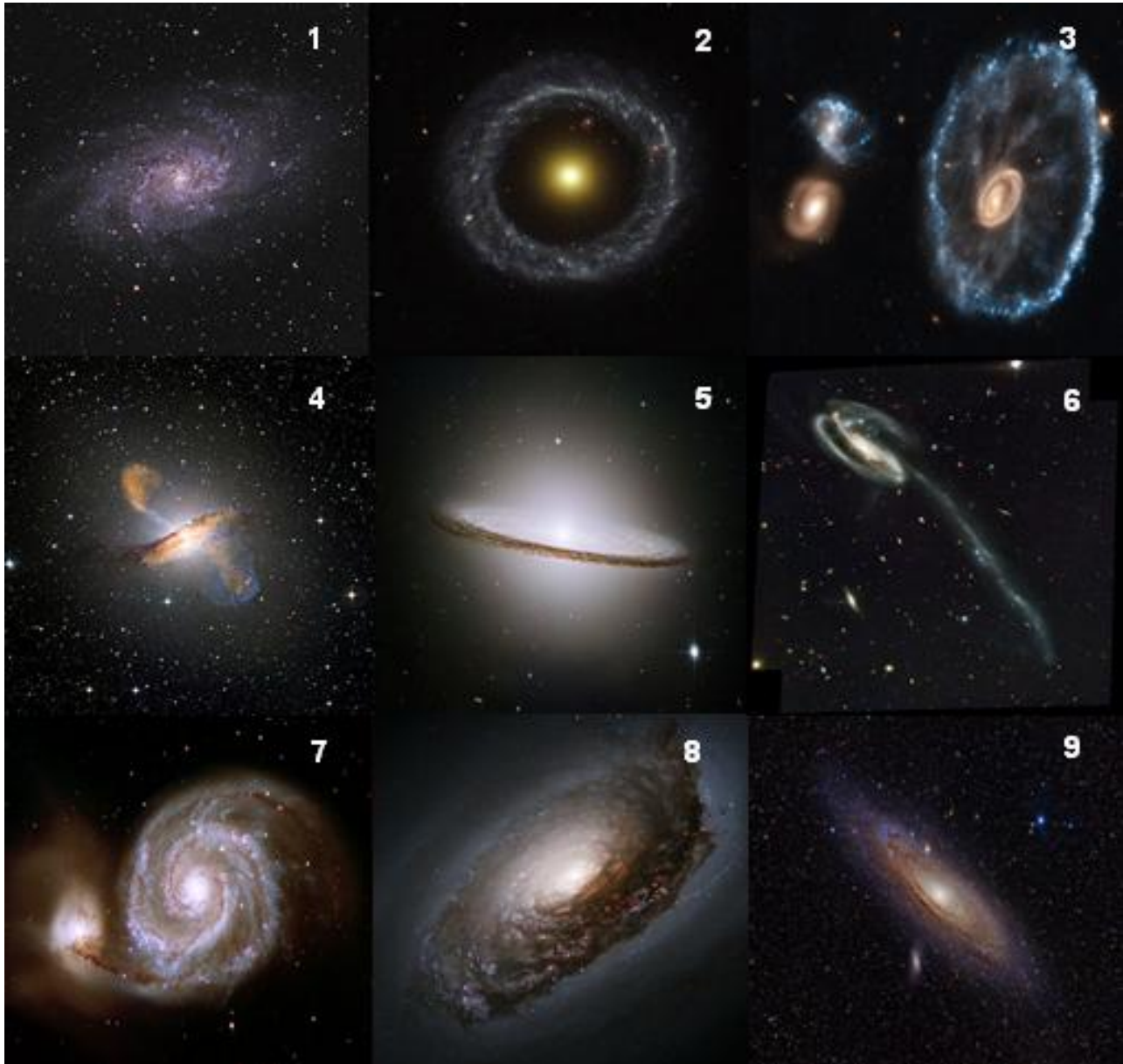
- i. What is the L.S.T. (hh:mm) at the culmination of the Sun on 53rd day after the longest day in the northern hemisphere? [4] – 0848
- ii. The axial tilt of Mars. (integer) [2] – 25
- iii. How many images of the same distant quasar appear at the Einstein Cross? [1] – 4
- iv. mass of the Pluto in kg (exponent of 10 only) [2] – 22
- v. Put in the right order the meteor showers according to when their peak occurs, starting from the earlier one: [4] – 4312
 - 1.Perseids
 - 2.Orionids
 - 3.Eta Aquarids
 - 4.Quadrantids

M.

- i. Two white dwarfs have the same effective temperature. The one dwarf has an Absolute Bolometric Magnitude of $M_{bolA}=10.5mag$ and a mass $m_A=1$ solar mass, while the other has $M_{bolB}=10mag$. What is the second dwarf's mass in solar masses? The mass-radius relationship of a white dwarf is $R \sim 1/M$. (one decimal) [2] – 05
- ii. The value of the Constant of Aberration (arcseconds). (one decimal) [3] – 205
- iii. Because of the solar parallax, the time the Sun stays below the horizon of the North Pole is longer. How many minutes is this increase in time? (integer) [2] – 18
- iv. How many billions of years is the age of the Universe? (one decimal) [3] – 138
- v. What is the maximum ecliptic latitude that can be acquired by Pluto? Pluto's aphelion is $49.30AU$, its perihelion is $29.58AU$ and its orbital inclination is 17.17° . Earth's aphelion is $1.017AU$ and its perihelion is $0.983AU$. (one decimal) [3] – 178

N.

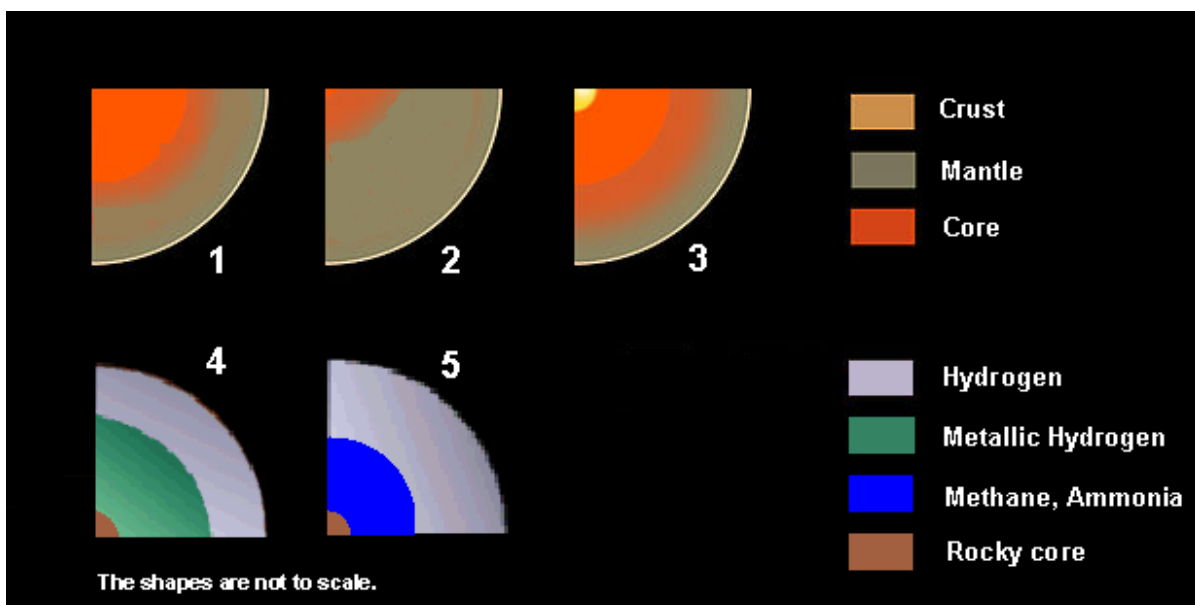
- i. By how many magnitudes will the magnitude of the faintest stars that can be detected by a CCD increase, if the exposure time is doubled? (up to two decimals) [3] – 075
- ii. Sombrero Galaxy, Tadpole Galaxy, Black Eye Galaxy, Hoag's Object [4] – 5682



- iii. How many flavours do the quarks have? [1] – 6
- iv. How many colours do the quarks have? [1] – 3
- v. A star with R.A. = $17^{\text{h}} 8^{\text{m}}$ rises in the sky at L.S.T. = $5^{\text{h}} 31^{\text{m}}$. How long the star will stay above the horizon (hh:mm)? [4] – 2314

O.

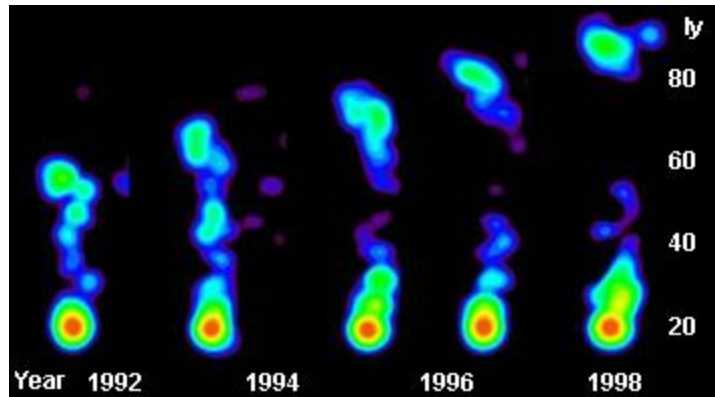
- i. The maximum effective temperature of the surface of a Cepheid is $9,000K$ and the minimum is $7,000K$. The difference between its brightness maximum and minimum is 2.0 Absolute Bolometric Magnitudes. How many times bigger is the maximum radius of the Cepheid than its minimum radius? (one decimal) [2] – 15
- ii. In 2013, after how many days after the Summer Solstice did the Earth reach Aphelion? (integer) [2] – 14
- iii. The Sun belongs to the Population _ stars. [1] – 1
- iv. The comet LINEAR is a periodic comet with aphelion distance of 5.29 A.U. And aphelion velocity as 10.45km/s . What is the semi-major axis of its orbit in A.U.? (up to 1st decimal) [2] – 39
- v. Identify the following planets by recognizing their internal structure: Jupiter, Neptune, Mercury [3] – 451



- vi. A main sequence star “fuses” Hydrogen at a rate of 1.178×10^{12} kg/s. Its luminosity (in 10^{24} W) if mass-defect is only 0.007. (integer) [3] – 742

P.

- i. What is the Azimuth of the point $\mathcal{V}$ (degrees) at the 18th Sidereal Hour? (integer) [2] – 90
- ii. Shape of Analemma of the Sun as seen from the Earth [1] – 8
- iii. What is the distance of a galaxy (in Mpc) with recession velocity of 13966.8km/s? [3] – 206
- iv. The apparent speed of this active galaxy's jet is $3.6c$! If the angle of the jet with the line of sight is 1.5° , what is the true speed of the jet? (up to 3rd decimal) [4] – 0993

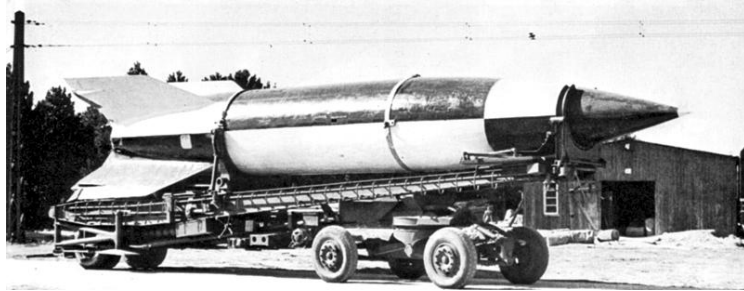


- v. On the 21st of March, at a place with latitude $\phi = 35^\circ$, during the sunset, a star on the ecliptic is at its upper culmination. What is its zenith distance in degrees? (one decimal) [3] – 115

Q.

i. $294^{\circ} 30'$ in radians [3] – 514

ii. The first photo from space was taken from a V-2. [1] – 2



iii, A mission is sent to Mars by following a Hohmann-Vetchinkin orbit. How many days is the minimum time interval that the mission's members will have to stay on Mars before they find the first opportunity to come back to Earth by following once more a Hohmann-Vetchinkin orbit? Consider that the orbits of Earth and Mars are circular and coplanar and that the distance of Mars from the Sun is $1.52AU$. (one decimal) [4] – 4586

iv. Mass of the supermassive black hole at the galactic centre in kg (exponent of 10 only) [2] – 37

v. distance of globular cluster M68 (in kpc) given its parallax = $97.2 \mu\text{arcsec}$ [3] – 103