Problem A.1: Interstellar Mission (4 Points)

You are on an interstellar mission from the Earth to the 8.7 light-years distant star Sirius. Your spaceship can travel with 70% the speed of light and has a cylindrical shape with a diameter of 6 m at the front surface and a length of 25 m. You have to cross the interstellar medium with an approximated density of 1 hydrogen atom/m^3.

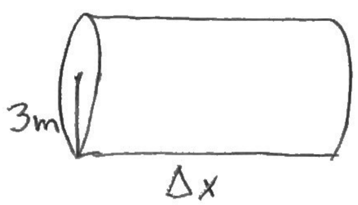
(a) Calculate the time it takes your spaceship to reach Sirius.
(b) Determine the mass of interstellar gas that collides with your spaceship during the mission.

Note: Use 1.673×10^{-27} kg as proton mass.

a) Ignoring drag, gravitational atraction and radiation pressure, we can assume that the speed is constant, thus:

$$T = \frac{\Delta x}{v} \quad , \quad v = 0{,}7 \cdot 3 \cdot 10^8 \text{ m/s} \quad \Delta x = 8.7 \cdot 365{,}25 \cdot 24 \cdot 3600 \cdot c \Rightarrow \boxed{T \cong 3{,}9 \cdot 10^8 \text{ s}}$$

b) The volume V swepted by the spaceship is:



$$V = 9\pi \Delta x \quad , \quad \rho = \frac{m}{V} \Rightarrow m = \rho \cdot V, \quad \rho = 1{,}673 \cdot 10^{-27} \text{ Kg}$$

$$\Rightarrow \boxed{m \cong 1{,}1 \cdot 10^{-7} \text{ Kg}}$$

Problem A.2: Time Dilation (4 Points)

Because you are moving with an enormous speed, your mission from the previous problem A.1 will be influenced by the effects of time dilation described by special relativity: Your spaceship launches in June 2020 and returns back to Earth directly after arriving at Sirius.

(a) How many years will have passed from your perspective?
(b) At which Earth date (year and month) will you arrive back to Earth?

a) $T = 3{,}9 \cdot 10^8 s$

From my perspective would have passed Δt years:

$\Delta t = 2T \cdot \sqrt{1 - \frac{v^2}{c^2}} \cong 5.56 \cdot 10^8 s \Rightarrow \boxed{\Delta t = 17{,}8 \text{ years}}$

b) June 2020 + 2T ; $2T \cong 24{,}86$ years $\Rightarrow$ 24 years and 10 months

Thus I would arrive back to Earth at $\boxed{\text{April 2045}}$.

Problem A.3: Magnitude of Stars (4 Points)

The star Sirius has an apparent magnitude of -1.46 and appears 95-times brighter compared to the more distant star Tau Ceti, which has an absolute magnitude of 5.69.

(a) Explain the terms *apparent magnitude*, *absolute magnitude* and *bolometric magnitude*.
(b) Calculate the apparent magnitude of the star Tau Ceti.
(c) Find the distance between the Earth and Tau Ceti.

a) Apparent magnitude: scale that compares the apparent brightness of the stars, so it depends on the distance to the observer and the luminosity of the star.
Absolute magnitude: scale that compares the brightness of the stars if all of them were at 10pc from the observer, so it only compares their luminosity (intrinsic caracteristic).
Bolometric magnitude: magnitude integrated over all wavelenghts.

b) Pogson's equation:

$$m_S - m_\tau = -2,5 \log\left(\frac{f_S}{f_\tau}\right), \frac{f_S}{f_\tau} = 95 \Rightarrow \boxed{m_\tau \cong 3.48}$$

c)

$$m_\tau - M_\tau = 5 \log d(pc) - 5 \Rightarrow d = 10^{\frac{m_\tau - M_\tau + 5}{5}} \Rightarrow \boxed{d \cong 3.62 pc}$$

Problem A.4: Emergency Landing (4 Points)

Because your spaceship has an engine failure, you crash-land with an emergency capsule at the equator of a nearby planet. The planet is very small and the surface is a desert with some stones and small rocks laying around. You need water to survive. However, water is only available at the poles of the planet. You find the following items in your emergency capsule:

- Stopwatch
- Electronic scale
- 2m yardstick
- 1 Litre oil
- Measuring cup

Describe an experiment to determine your distance to the poles by using the available items.

Hint: *As the planet is very small, you can assume the same density everywhere.*

First we can pick up a small rock and measure, with the eletronic scale, its mass m. Then putting the rock inside the measuring cup with an initial volume V_0 of oil, with the rock the volume would go up to V, thus $\Delta V = V - V_0$ is the volume of the rock. Since the density is the same everywhere, the density of the planet is equal to the density of the rock $\rho = \frac{m}{\Delta V}$. After that, again with the rock and with the 2m yardstick, if we release the rock at the top of the yardstick and measure the time Δt it takes to reach the floor we could calculate the gravity acceleration $g = \frac{GM}{R^2}$: $2 = \frac{g\Delta t^2}{2} \Rightarrow g = \frac{4}{\Delta t^2} \Rightarrow R^2 = \frac{GM\Delta t^2}{4}$. Since $M = \frac{4}{3}\pi R^3 \rho \Rightarrow R = \frac{3}{\pi \rho G \Delta t^2}$. The distance to the pole $d = \frac{\pi}{2} R$ will be:

$$d = \frac{3}{2 G \rho \Delta t^2}, \quad \rho = \frac{m}{\Delta V} \Rightarrow \boxed{d = \frac{3 \Delta V}{2 G m \Delta t^2}}$$

Problem B.1: Temperature of Earth (6 Points)

Our Sun shines bright with a luminosity of 3.828×10^{26} Watt. Her energy is responsible for many processes and the habitable temperatures on the Earth that make our life possible.

(a) Calculate the amount of energy arriving on the Earth in a single day.
(b) To how many litres of heating oil (energy density: 37.3×10^6 J/litre) is this equivalent?
(c) The Earth reflects 30% of this energy: Determine the temperature on Earth's surface.
(d) What other factors should be considered to get an even more precise temperature estimate?

Note: The Earth's radius is 6370 km; the Sun's radius is 696×10^3 km; 1 AU is 1.495×10^8 km.

a) $P = f \cdot A = \frac{L_\odot}{4\pi d^2} \cdot \pi R_\oplus^2 \Rightarrow P \cong 1.737 \cdot 10^{17}$ J/s

$\Rightarrow \boxed{P \cong 1.50 \cdot 10^{22} \text{ J/day}}$

b) $V = \frac{1.50 \cdot 10^{22}}{37.3 \cdot 10^6} \Rightarrow \boxed{V \cong 4.02 \cdot 10^{14} \text{ L}}$

c) Stefan-Boltzmann Law: ($a = 0.3$)

$L_\oplus = 4\pi R_\oplus^2 \sigma T_\oplus^4 \Rightarrow \frac{L_\odot}{4\pi d^2} \cancel{\pi} \cancel{R_\oplus^2} \cdot (1-a) = 4\cancel{\pi} \cancel{R_\oplus^2} \sigma T_\oplus^4$

$\Rightarrow T_\oplus = \left(\frac{L_\odot(1-a)}{16\pi\sigma d^2}\right)^{\frac{1}{4}} \cong 255 \text{ K} \Rightarrow \boxed{T_\oplus \cong 255 \text{ K}}$

d) We would also need to consider the atmospheric refraction, the extinction and the greenhouse effect.

Problem B.2: Distance of the Planets (6 Points)

The table below lists the average distance R to the Sun and orbital period T of the first planets:

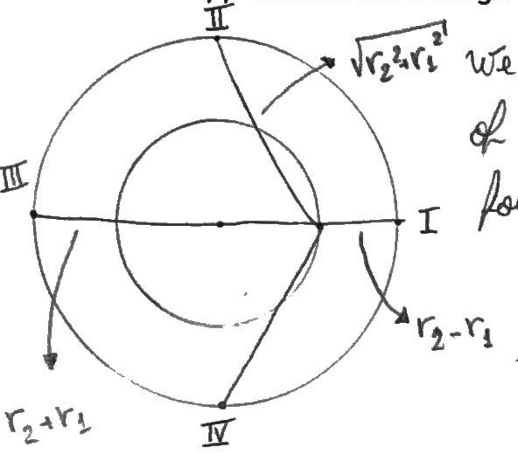
	Distance	Orbital Period
Mercury	0.39 AU	88 days
Venus	0.72 AU	225 days
Earth	1.00 AU	365 days
Mars	1.52 AU	687 days

(a) Calculate the average distance of Mercury, Venus and Mars to the Earth.
Which one of these planets is the closest to Earth on average?

(b) Calculate the average distance of Mercury, Venus and Earth to Mars.
Which one of these planets is the closest to Mars on average?

(c) What do you expect for the other planets?

Hint: *Assume circular orbits and use symmetries to make the distance calculation easier. You can approximate the average distance by using four well-chosen points on the planet's orbit.*



We can find the average distance by finding the average of the distances between the inner planet and the four points I, II, III and IV:

$$\langle r \rangle \cong \frac{r_2+r_1+\sqrt{r_2^2+r_1^2}+r_2-r_1+\sqrt{r_2^2+r_1^2}}{4}$$

$$\Rightarrow \langle r \rangle \cong \frac{r_2+\sqrt{r_2^2+r_1^2}}{2}$$

a) Earth x Mercury : $\langle r_1 \rangle \cong 1.037$ U.A.
x Venus : $\langle r_2 \rangle \cong 1.116$ U.A.
x Mars : $\langle r_3 \rangle \cong 1.670$ U.A.
$\Rightarrow$ Mercury is the closest on average

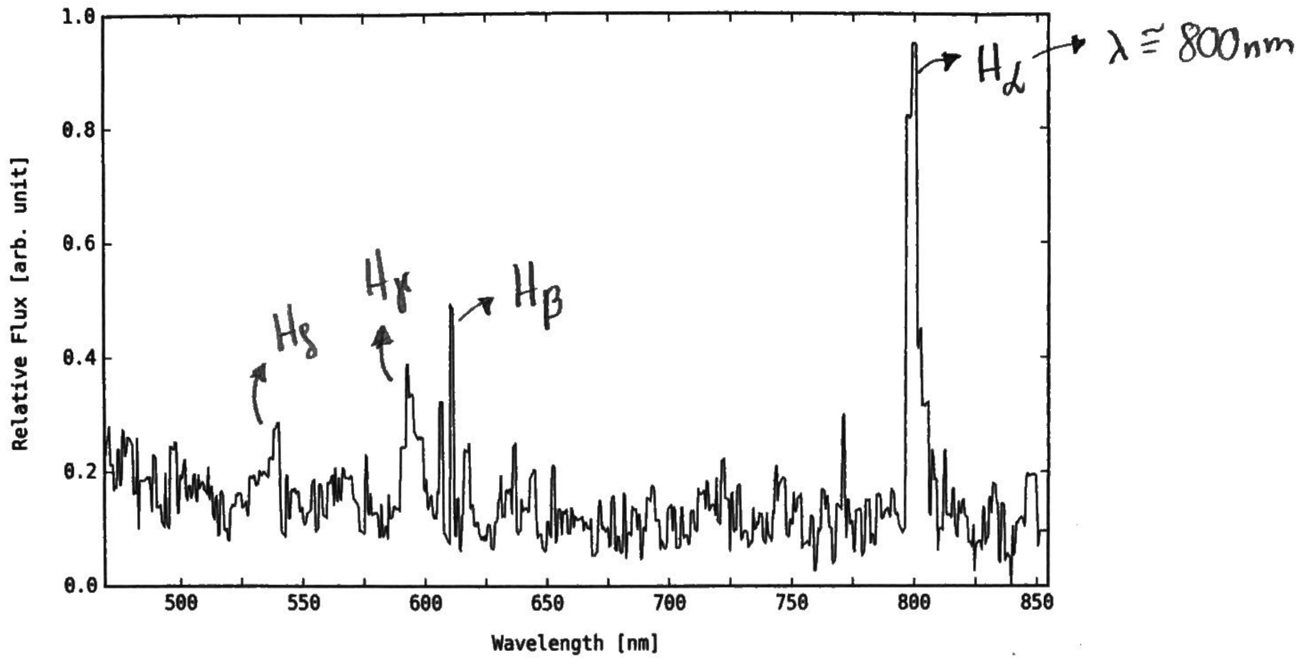
b) Mars x Mercury : $\langle r_1' \rangle \cong 1.545$ U.A.
x Venus : $\langle r_2' \rangle \cong 1.601$ U.A.
x Earth : $\langle r_3' \rangle \cong 1.670$ U.A.
$\Rightarrow$ Mercury is the closest on average

c) Is expected that Mercury is the closest planet to all other planets, on average. Because for a planet with distance R, an outer planet would be farther than an inner one, because $\frac{r_2+\sqrt{r_2^2+R^2}}{2} > \frac{R+\sqrt{R^2+r_1^2}}{2}$, $r_2 > R > r_1$

Then, the closest planet on average will be the one with the smallest r_1.

Problem B.3: Mysterious Object (6 Points)

Your research team analysis the light of a mysterious object in space. By using a spectrometer, you can observe the following spectrum of the object. The Hα line peak is clearly visible:



(a) Mark the first four spectral lines of hydrogen (Hα, Hβ, Hγ, Hδ) in the spectrum.
(b) Determine the radial velocity and the direction of the object's movement.
(c) Calculate the distance to the observed object.
(d) What possible type of object is your team observing?

b) $\lambda_\alpha \cong 660$ nm
$\lambda \cong 800$nm $\Rightarrow \frac{\lambda}{\lambda_\alpha} = \sqrt{\frac{1+\frac{v_r}{c}}{1-\frac{v_r}{c}}} \Rightarrow v_r = c \cdot \frac{\left(\frac{\lambda}{\lambda_\alpha}\right)^2 - 1}{\left(\frac{\lambda}{\lambda_\alpha}\right)^2 + 1} \Rightarrow \boxed{v_r \cong 5.7 \cdot 10^7 \text{ m/s}}$

Since $v_r > 0$ the object is moving away from Earth.

c) Hubble law: $v_r = H_0 \cdot d$, $H_0 \cong 70 (\text{Km/s})/\text{Mpc} \Rightarrow \boxed{d \cong 814 \text{ Mpc}}$

d) Since it is very far away and has a very high redshift it is probably a quasar.

Problem B.4: Distribution of Dark Matter (6 Points)

The most mass of our Milky Way is contained in an inner region close to the core with radius R_0. Because the mass outside this inner region is almost constant, the density distribution can be written as following (assume a flat Milky Way with height z_0):

$$\rho(r) = \begin{cases} \rho_0, & r \leq R_0 \\ 0, & r > R_0 \end{cases}$$

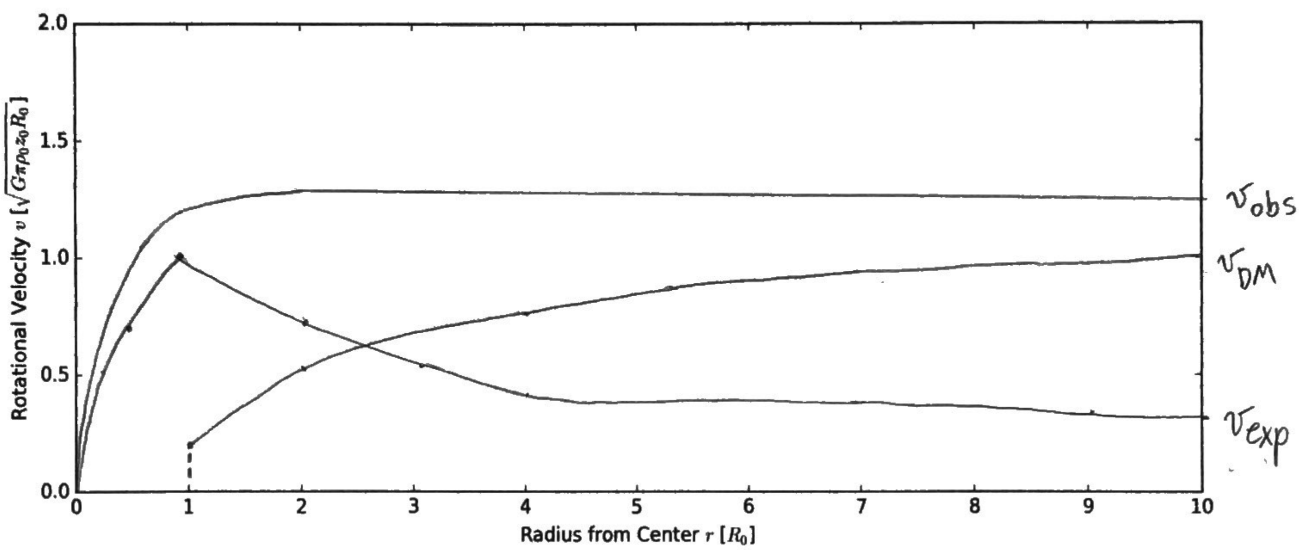
(a) Derive an expression for the mass $M(r)$ enclosed within the radius r.

(b) Derive the expected rotational velocity of the Milky Way $v(r)$ at a radius r.

(c) Astronomical observations indicate that the rotational velocity follows a different behaviour:

$$v_{obs}(r) = \sqrt{G\pi\rho_0 z_0 R_0}\left(\frac{5/2}{1+e^{-4r/R_0}} - \frac{5}{4}\right)$$

Draw the expected and observed rotational velocity into the plot below:



(d) Scientists believe the reasons for the difference to be *dark matter*: Determine the rotational velocity due to dark matter $v_{DM}(r)$ from R_0 and draw it into the plot above.

(e) Derive the dark matter mass $M_{DM}(r)$ enclosed in r and explain its distributed.

(f) Explain briefly three theories that provide explanations for *dark matter*.

a) $r \leqslant R_0 : M(r) = \pi r^2 z_0 \cdot \rho_0$

$r > R_0 : M(r) = \pi R_0^2 z_0 \rho_0 \Rightarrow$ $\boxed{M(r) = \begin{cases} \pi z_0 \rho_0 r^2, & r \leqslant R_0 \\ \pi z_0 \rho_0 R_0^2, & r > R_0 \end{cases}}$

b) $v = \sqrt{\frac{GM}{r}} \Rightarrow \boxed{v(r) = \begin{cases} \sqrt{G\pi z_0 \rho_0 r}, & r \leqslant R_0 \\ \sqrt{\frac{G\pi z_0 \rho_0 R_0^2}{r}}, & r > R_0 \end{cases}} \Rightarrow \begin{matrix} \sqrt{r} \cdot \sqrt{G\pi z_0 \rho_0} \\ \frac{R_0}{\sqrt{r}} \cdot \sqrt{G\pi z_0 \rho_0} \end{matrix}$

c) $y = \frac{v}{\sqrt{G\pi z_0 \rho_0 \cdot R_0}}, \quad x = \frac{r}{R_0} \Rightarrow y_1 = \begin{cases} \sqrt{x}, & r \leqslant R_0 \\ \frac{1}{\sqrt{x}}, & r > R_0 \end{cases}$

$y_2 = \frac{5}{4}\left(\frac{2}{1+e^{-4x}} - 1\right)$

d) $y_3 \cong \frac{5}{4} - \frac{1}{\sqrt{x}}, \; r > R_0 \Rightarrow v_{DM}(r) \cong \sqrt{G \cdot \pi z_0 \rho_0 R_0} \cdot \left(\frac{5}{4} - \frac{1}{\sqrt{x}}\right)$

e) $\sqrt{\frac{G(M+M_{DM})}{r}} \cong \sqrt{\frac{GM}{R_0}} \cdot \left(\frac{5}{4} - \frac{1}{\sqrt{x}}\right) \Rightarrow M + M_{DM} = M \cdot x \left(\frac{5}{4} - \frac{1}{\sqrt{x}}\right)^2$

$M_{DM} = M \cdot \left[\left(\frac{5}{4}\sqrt{x} - 1\right)^2 - 1\right]$

$\Rightarrow M_{DM} = M\left(\frac{5}{4}\sqrt{x}\right)\left(\frac{5}{4}\sqrt{x} - 2\right) = M\left(\frac{25}{16}x - \frac{5}{2}\sqrt{x}\right)$

$\Rightarrow \boxed{M_{DM} = \sqrt{\pi z_0 \rho_0 R_0^2} \cdot \left(\frac{25}{16}\frac{r}{R_0} - \frac{5}{2}\sqrt{\frac{r}{R_0}}\right)} \Rightarrow$ the dark matter mass tends to increase with the distance from the center.

f) • Dark matter does not exist, we only need a reformulation in the general theory of relativity.

• Entropic gravity: gravity is an entropic force, thus it is the result of a series of probabilistic events.

• MOND: Newton's laws are violated for small forces, so a new law that consider the movement of galaxies must be found.

Problem C.1 : Detection of Gravitational Waves (10 Points)

This problem requires you to read the following recently published scientific article:

Observation of Gravitational Waves from a Binary Black Hole Merger.
B. P. Abbott et al., LIGO Scientific Collaboration and Virgo Collaboration
arXiv:1602.03837, (2016). Link: https://arxiv.org/pdf/1602.03837.pdf

Answer following questions related to this article:

(a) How was the existence of gravitational waves first shown?

Albert Einstein predicted its existence as a solution to the linearized weak field equations and it was first observed by LIGO, where it detectors observed two black holes merging

(b) Which detectors exist around the world? Why did only LIGO detect GW150914?

TAMA 300 in Japan, GEO 600 in Germany, LIGO in the USA and Virgo in Italy. Only LIGO detected it, because only them were observing at that time. Virgo was upgrading and GEO 600 was not sufficiently sensitive and it was not in observational mode.

(c) Explain the components of the LIGO detectors.

It has two arms, each with two mirrors separated by 4Km. The arms are perpendicular, so the phase of the light emitted by the laser is different in each arm if a gravitational wave go through it, altering the arm lengths.

(d) Describe the different sources of noise. How was their impact reduced?

Photon shot noise, seismic noise and thermal noise. They can be reduced by maximizing the conversion of strain to optical signal, isolating the test masses, suspending them in a quadruple-pendulum system, using low-mechanical-loss materials in the test masses and their suspensions, components mounted on vibration isolation stages in ultrahigh vacuum etc..

(e) What indicates that the gravitational wave originated from the merger of a black hole?

A lot of detectors in LIGO were used that show that the GW150914 was not an instrument artifact or a noise. Also because of the frequency and the amplitude of the wave, there are a lot of evidences that corroborates with the black hole merge theory.

(f) Which are the methods used to search for gravitational wave signals in the detector data?

Generic transient search: operates without a specific waveform model, it identifies coincident excess power in time-frequency representations.

Binary coalescence search: targets gravitational-wave emission from binary systems with total mass less than 100 $M_\odot$.

(g) How were the source parameters (mass, distance, etc.) determined from the data?

Using relativity-based models and fitting the data in numerical simulations of binary black holes mergers.

Problem C.2 : First Image of a Black Hole (10 Points)

This problem requires you to read the following recently published scientific article:

First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole.
The Event Horizon Telescope Collaboration, arXiv:1906.11238, (2019). Link: https://arxiv.org/pdf/1906.11238.pdf

Answer following questions related to this article:

(a) Calculate the photon capture radius and the Schwarzschild radius of M87* (in AU).

$M \cong 6.2 \cdot 10^9 M_\odot$ $\Rightarrow$ $R_S = \frac{2GM}{c^2}$ $\Rightarrow$ $\boxed{R_S \cong 122 \text{ U.A.}}$

$M_\odot \cong 1.99 \cdot 10^{30} Kg$ $R_C = \sqrt{27}\frac{GM}{c^2}$ $\boxed{R_C \cong 318 \text{ U.A.}}$

(b) Why was it not possible for previous telescopes to take such a picture of the black hole?

They couldn't resolve it and weren't sensitive enought.

(c) Describe the components and functionality of the event horizon telescope.

It is an array of radio telescopes with an effective diameter of almost Earth, thus it has a very low resolution angle. It uses interferometry (VLBI) to reach the caracteristics of a Earth's diameter telescope.

(d) Explain the two algorithms used to reconstruct the image from the telescope data.

CLEAN: inverse-modeling approach that deconvolves the interferometer point-spread function from the Fourier-transformed visibilities.

RML: forward-modeling approach that searches for an image that is not only consistent with the observed data but also favors specified image properties.

(e) What parameters were required for the GRMHD simulations to generate an image?

Dimensionless spin $a_* \equiv Jc / GM^2$ and the dimensionless magnetic flux over the event horizon $\phi = \Phi \cdot (\dot{M} R_g^2)^{\frac{1}{2}}$

(f) Explain the physical origins of the features in Figure 3 (central dark region, ring, shadow).

Central dark region: is the event horizon, where not even the light can escape.
Ring: photons that are captured by an unstable circular orbit (strong gravitational lensing).
Shadow: area enclosed by the photon capture radius.

(g) How can the image resolution be increased in future observations?

Using a shorter wavelength, i.e., 0.8 mm, adding more telescopes and by doing space based interferometry