Solutions and Marking Scheme for the Specimen Paper

by 6th SAO Organising Committee

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Should you find any errors in the solutions, please direct your query to sao@astronomy.sg.

Solutions – Part A

(Q1) Model white dwarf **[2 marks]**

White dwarves are stellar core remnants that resist gravitational collapse primarily through electron degeneracy pressure. A non-relativistic calculation gives rise to a polytrope solution, that describes the relation between the pressure and density of a model white dwarf,

$$P = K_1 \rho^{\frac{5}{3}}$$

Express the units of the proportionality constant K_1 in SI units.

Solution:

Rearrange the equation to get K_1:

$$K_1 = \frac{P}{\rho^{\frac{5}{3}}}$$

Rewrite pressure and density as their definitions:

$$K_1 = \frac{\frac{force}{area}}{\left(\frac{mass}{volume}\right)^{5/3}}$$

$$= \frac{mass \times acceleration}{area \times \left(\frac{mass}{volume}\right)^{5/3}}$$

$$= \frac{acceleration \times volume^{5/3}}{area \times mass^{2/3}}$$

$$= \frac{[m\ s^{-2}][m^3]^{5/3}}{[m^2][kg]^{2/3}}$$

$$= [\boldsymbol{m^4\ s^{-2}\ kg^{-2/3}}]$$

2m

(Q2) Horsepower **[2 marks]**

The metric horsepower (units: PS) was briefly used as a power-measuring unit in the 1960s and retains specific use today to describe the power output of automobile engines. 1 PS was defined as the power required to raise a mass of 75 kilograms against the Earth's gravitational force over a distance of one metre in one second. Express the luminosity of the Sun, in metric horsepower.

Solution:

Starting with the definition of power,

$$P = \frac{W}{t} = \frac{F\Delta h}{t} = \frac{mg\Delta h}{t}$$
$$= \frac{(75\ kg)(9.81\ m\ s^{-2})(1\ m)}{(1\ s)}$$
$$\Rightarrow 1\ hp = 735.75\ W$$

Hence, the luminosity of the Sun is

$$L_{Sun} = \frac{3.828 \times 10^{26}\ W}{735.75\ W\ hp^{-1}} = \mathbf{5.20 \times 10^{23}\ hp}$$ **2m**

(Q3) Hot and hotter **[2 marks]**

Sirius is a binary star system that is the brightest star in the Earth's night sky. Estimate the ratio of the surface temperatures of Sirius A to that of Sirius B.

Solution:

Recall that Sirius A is a main sequence star about twice as massive as the Sun and Sirius B is a white dwarf of similar mass to our Sun.

We can thus infer that,

(1) Being a white dwarf, Sirius B has a higher temperature than Sirius A, i.e. ratio < 1, where $T_B \sim 25000\ K$

(2) Sirius A should have a higher temperature than our main-sequence Sun, trivially around $T_A \sim 2(5780) = 11600\ K$

Given the actual surface temperatures of Sirius A and B are 9940 K and 25200 K respectively, the answer is

$$Ratio = \frac{T_A}{T_B} = \frac{9940\ K}{25200\ K} = \mathbf{0.394}$$ **2m**

Answers between 0.150 and 0.800 inclusive score full credit.

(Q4) M1 **[2 marks]**

M1, the *Crab Nebula*, is the supernova remnant associated with the bright supernova observed by ancient astronomers in year 1054. Estimate the largest angular dimension of M1, in arcseconds.

Solution:

Recall that M1 is a small object that is difficult to find in low-power telescope setups, hence its maximum angular size is likely to be around a few arcminutes, i.e. in the hundreds of arcseconds.

The actual size of M1 is modelled by an ellipse with angular semi-major and semi-minor axes of

$$a = 420'' , b = 290''$$

Hence its largest angular dimension is $\mathbf{420''}$. **2m**

Answers between $\mathbf{100''}$ and $\mathbf{800''}$ inclusive score full credit.

(Q5) The dieting Sun **[2 marks]**

Kallenrode (2004) estimated that the rate of particles being carried away from the Sun by the solar wind is $n = 1.3 \times 10^{36}/s$. The Sun's mass also concurrently decreases due to nuclear fusion of hydrogen in its core. Estimate the ratio of the mass loss rates from the solar wind to that of nuclear fusion.

Solution:

The mass loss rate from nuclear fusion can be computed from the Sun's luminosity using Einstein's mass-energy equivalence relation, $E = mc^2$,

$$\Delta M_{fusion} = \frac{L}{c^2}$$

The solar wind primarily consists of protons and electrons due to their light mass and charge. Observe that for the Sun's electrical charge to remain unchanged (slightly positive) over long periods of time, the solar wind must be electrically neutral, i.e. equal numbers of protons and electrons must be lifted off the surface into the solar wind.

Then, the mass loss rate from solar wind is approximately

$$\Delta M_{solar\,wind} = \frac{n}{2}(m_{proton} + m_{electron})$$

Hence, the ratio of the mass loss rates is

$$
\begin{aligned}
R &= \frac{nc^2\left(m_{proton} + m_{electron}\right)}{L} \\
&= \frac{(1.3 \times 10^{36})(2.998 \times 10^{8})^2(1.673 \times 10^{-27} + 9.109 \times 10^{-31})}{2(3.828 \times 10^{26})} \\
&= \mathbf{0.255} \approx \mathbf{\frac{1}{4}}
\end{aligned}
$$

2m

Answers between 0.1 and 1 inclusive score full credit.

(Q6) And what about the Earth? **[2 marks]**

Indicate if the following statement is True or False.

≫ Assuming the mass of the rest of the Solar System remains unchanged, the semi-major axis of Earth's orbit increases as the Sun evolves towards a red giant.

Solution:

As the Sun evolves towards a red giant, both nuclear fusion and solar wind will reduce the mass of the Sun and hence lead to a decrease in the force of gravity between the Sun and the planets.

Furthermore, as the Sun experiences mass loss, the gravitational potential energy of a planet increases (becomes less negative) whereas the kinetic energy remains unchanged, since the latter is intrinsic to the planet and the former is a property of the Sun-planet system. Hence the orbital energy and orbital angular momentum is conserved (for an isotropic solar wind) as the planets are raised to higher orbits.

Thus the statement is **True**, although the effect is in practice negligibly small and thus impossible to discern from the other sources perturbing planetary orbits. **2m**

Solutions – Part B

(Q7) The ear of grain **[9 marks]**

The star α *Virginis*, with common name *Spica*, is a binary star system, with a combined apparent V-band magnitude of $m_V = 0.97$. Interferometry of the binary components by Herbison-Evans et al. (1971) gives the absolute V-band magnitudes of the two components as $M_{V,1} = -3.50$ and $M_{V,2} = -1.50$. Calculate the distance to *Spica*.

Solution:

From the definition of absolute magnitude, comparing the two components, the ratio of their relative luminosities is

$$M_{V,2} - M_{V,1} = -2.5\lg\left(\frac{L_2}{L_1}\right)$$
$$\Rightarrow \frac{L_2}{L_1} = 10^{\frac{M_{V,1}-M_{V,2}}{2.5}}$$
$$= 10^{\frac{-3.50-(-1.50)}{2.5}}$$
$$= 0.1585$$

3m

Then, comparing the luminosities of the combined pair to that of the brighter component alone,

$$M_V - M_{V,1} = -2.5\lg\left(\frac{L_V}{L_1}\right)$$
$$= -2.5\lg\left(\frac{L_1 + L_2}{L_1}\right)$$
$$\Rightarrow M_V = M_{V,1} - 2.5\lg\left(1 + \frac{L_2}{L_1}\right)$$
$$= -3.50 - 2.5\lg(1 + 0.1585)$$
$$= -3.660$$

3m

Using Pogson's law and rearranging, we can find d_{Spica}, the distance to Spica,

$$m_V - M_V = 5\lg\left(\frac{d_{Spica}}{10\ pc}\right)$$
$$\Rightarrow d_{Spica} = 10 \times 10^{\frac{m_V - M_V}{5}}\ pc$$
$$= 10 \times 10^{\frac{0.97-(-3.660)}{5}}\ pc$$
$$= \mathbf{84.3\ pc}$$

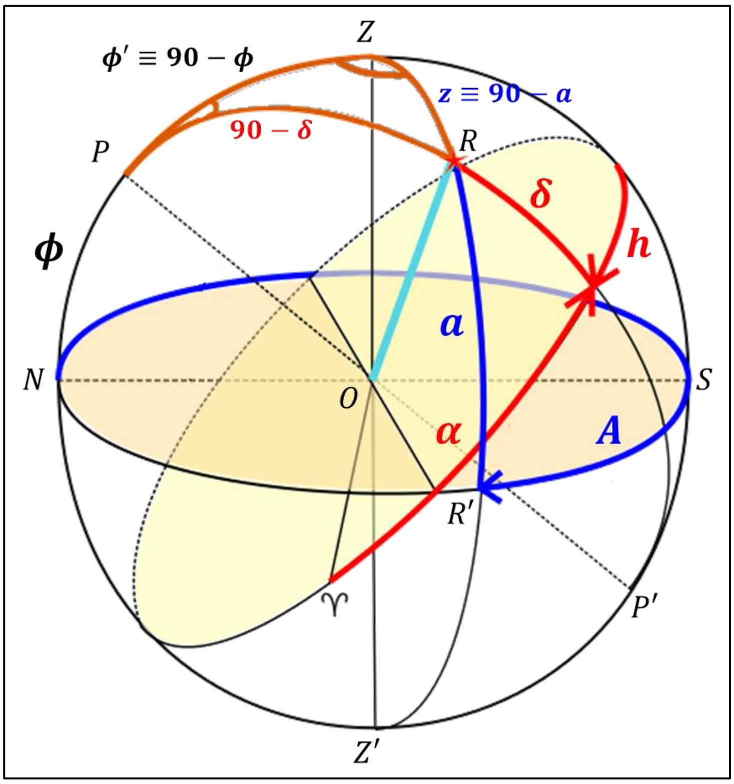
3m

(Q8) The solar eclipse of 9 Mar 2016 **[9 marks]**

A total solar eclipse occurred on 9 Mar 2016, although only a partial eclipse was visible from Singapore. From data provided by NASA (Espenak, 2014), the local time and local sidereal time at the point of local greatest eclipse was $08^h 24^m$ and $18^h 28^m$ respectively, at which the geocentric coordinates of the Sun were $(\delta, \alpha) = (-04°24', 23^h 19^m)$. Singapore has geographical coordinates of $(\phi, \lambda) = (1°17', 103°51')$ and is in the timezone UTC+08:00. Calculate the horizontal coordinates (altitude and azimuth) of the Sun at the point of greatest eclipse, (a, A) as seen from Singapore.

Solution:

The diagram below depicts the angles involved in the conversion from geocentric (equatorial) coordinates to horizontal coordinates.



First, we need to calculate the hour angle h (the angle between the Sun at greatest eclipse and the local meridian). With the local sidereal time, we have

$$\begin{aligned} h &= LST - \alpha \\ &= \left(\frac{360°}{24}\right)\left(18 + \frac{29}{60}\right) - \left(\frac{360°}{24}\right)\left(23 + \frac{19}{60}\right) \\ &= -72.75° \\ &= 287.25° \ (mod\ 360°) \end{aligned}$$

2m

Using the spherical cosine rule on PRZ, we can find the altitude,

$$\cos z = \cos\phi' \cos(90° - \delta) + \sin\phi' \sin(90° - \delta) \cos h$$
$$\sin a = \sin\phi \sin\delta + \cos\phi \cos\delta \cos h$$
$$\Rightarrow a = \arcsin(\sin\phi \sin\delta + \cos\phi \cos\delta \cos h)$$
$$= \arcsin(\sin 1.283° \sin(-4.4°) + \cos 1.283° \cos(-4.4°) \cos 287.25°)$$
$$= \mathbf{17.1}°$$

3m

Using the spherical cosine rule again on PRZ, we can find the azimuth,

$$\cos(90° - \delta) = \cos\phi' \cos z + \sin\phi' \sin z \cos(360° - A)$$
$$\sin\delta = \sin\phi \sin a + \cos\phi \cos a \cos A$$
$$\cos A = \frac{\sin\delta - \sin\phi \sin a}{\cos\phi \cos a}$$
$$\Rightarrow A = \arccos\left(\frac{\sin\delta - \sin\phi \sin a}{\cos\phi \cos a}\right)$$
$$= \arccos\left(\frac{\sin(-4.4°) - \sin 1.283° \sin 17.090°}{\cos 1.283° \cos 17.090°}\right)$$
$$= \mathbf{95.0}°$$

4m

Using the spherical sine rule on triangle PRZ will give $A = 85.0°$, the supplementary angle instead. No credit is given here, unless it is reasoned that, as the Sun's declination is negative, it rises due east of cardinal East and thus $(180° - A)$ is taken instead as the range of possible azimuths lies between $90° < A < 180°$.

(Q9) Model globular cluster **[9 marks]**

Consider a model globular cluster composed of N stars each of mass M, distributed uniformly within a circle of angular radius θ_c. Express the minimum aperture D required to resolve individual stars in this cluster when observing at a wavelength λ, in terms of the above quantities.

Solution:

For small angles, the solid angle subtended by the circle is approximately planar and thus

$$\Omega_c = \pi\theta_c^2$$

2m

Let the angular area of the cluster be partitioned amongst the N stars, each assumed to be at the centre of a smaller circle of radius θ and angular area Ω. This approximation improves as N grows larger. Then, this area is

$$\Omega = \frac{\Omega_c}{N} = \frac{\pi\theta_c^2}{N}$$

2m

The mean angular distance $\langle\theta\rangle$ between stars is then approximately the distance between the centres of two such smaller squares with $\Omega = \pi\theta^2$,

$$\begin{aligned}\langle\theta\rangle &= 2\theta \\ &= 2\sqrt{\frac{\Omega}{\pi}} \\ &= \frac{2\theta_c}{\sqrt{N}}\end{aligned}$$

2m

The resolution limit of an aperture, for small θ, is given by

$$\sin\theta \approx \theta = \frac{1.22\lambda}{D}$$

$$\Rightarrow D = \frac{1.22\lambda}{\theta}$$

2m

Hence, the minimum aperture D required is

$$\begin{aligned}D_{min} &= \frac{1.22\lambda}{\langle\theta\rangle} \\ &= \frac{1.22\lambda\sqrt{N}}{2\theta_c}\end{aligned}$$

1m

The above approximation for the mean angular distance is a factor of two larger than values generated from repeated simulations for $10 \leq N \leq 1000$. This implies the mean angular distance is closer to $\langle\theta\rangle = \theta$ instead.

All other methods of estimating the mean angular distance between stars in the cluster are awarded full credit, provided the assumptions are stated and the reasoning sound.

(Q10) Thin disk of the Milky Way **[9 marks]**

The thin disk is a key structural component found in both spiral and lenticular galaxies. Analysis and modelling of data from the Two Micron All-Sky Survey (2MASS) by Ojha (2000) provides estimates of the scale height and scale length of the Milky Way's thin disk at $h_z = 250\,pc$ and $h_r = 2800\,pc$ respectively. A paper by Karim et al. (2016) calculates that the Sun is at a distance above the Galactic mid-plane of $z_{Sun} = 17\,pc$. Using these estimates and the constants sheet, determine the Galactocentric distance r' along the mid-plane, at which the density of the disk has decreased to half the value from the density measured surrounding the Sun.

Solution:

Along the mid-plane ($z = 0$) of the Milky Way, the density at some Galactocentric distance r, decreases with distance from the centre ($r = 0$) as

$$\rho_r = \rho_{r=0} e^{-\frac{r}{h_r}}$$ **2m**

At a constant Galactocentric distance r, the density at a distance z relative to a reference at $z = 0$ is

$$\rho_z = \rho_{z=0} e^{-\frac{z}{h_z}}$$ **2m**

Combining the two equations above, the thin disk density at some Galactocentric distance r and mid-plane distance z is

$$\rho_{r,z} = \rho_{0,0} e^{-\left(\frac{r}{h_r}\right)-\left(\frac{z}{h_z}\right)}$$

To find the Galactocentric distance r' along the mid-plane ($z = 0$) where the density has decreased to half that around the Sun, substituting we equate

$$\frac{\rho_{Sun}}{\rho_{r',0}} = \frac{\rho_{0,0} e^{-\left(\frac{r_{Sun}}{h_r}\right)-\left(\frac{z_{Sun}}{h_z}\right)}}{\rho_{0,0} e^{-\left(\frac{r'}{h_r}\right)-\left(\frac{0}{h_z}\right)}} = 2$$

$$\Rightarrow e^{-\left(\frac{r_{Sun}}{h_r}\right)-\left(\frac{z_{Sun}}{h_z}\right)} = 2e^{-\left(\frac{r'}{h_r}\right)}$$ **2m**

Taking ln on both sides, rearranging and solving for r', we have

$$-\left(\frac{r_{Sun}}{h_r}\right) - \left(\frac{z_{Sun}}{h_z}\right) = \ln 2 - \left(\frac{r'}{h_r}\right)$$

$$\frac{r'}{h_r} = \ln 2 + \frac{r_{Sun}}{h_r} + \frac{z_{Sun}}{h_z}$$

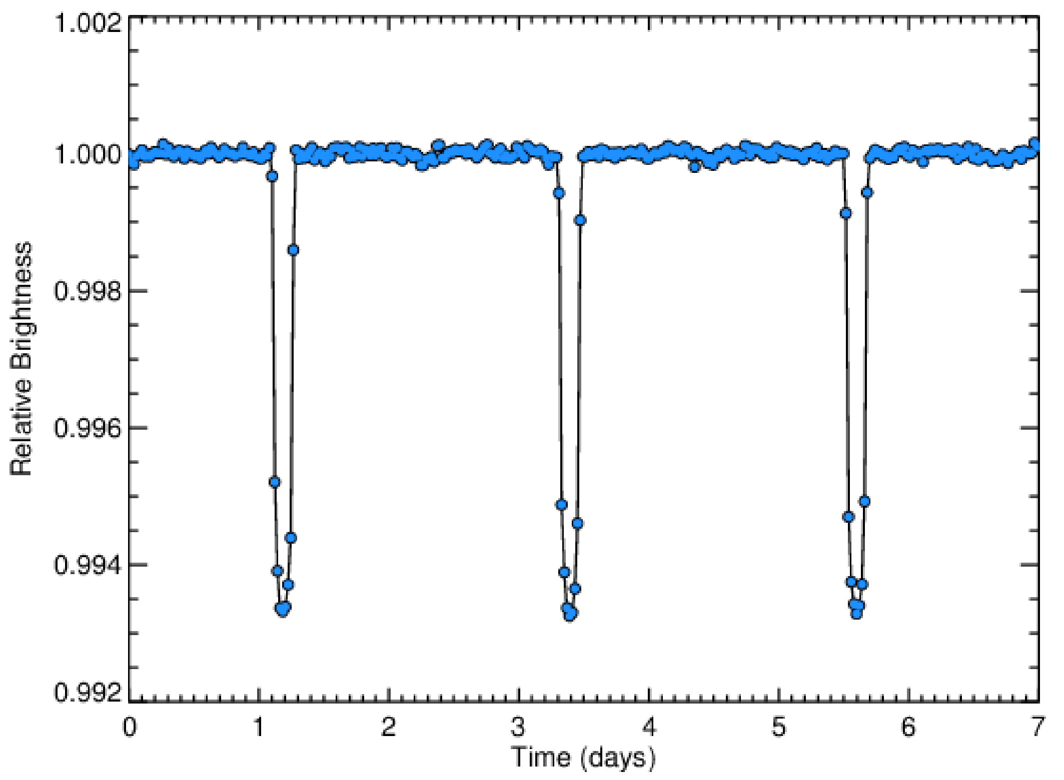
$$\Rightarrow r' = r_{Sun} + h_r\left(\ln 2 + \frac{z_{Sun}}{h_z}\right)$$

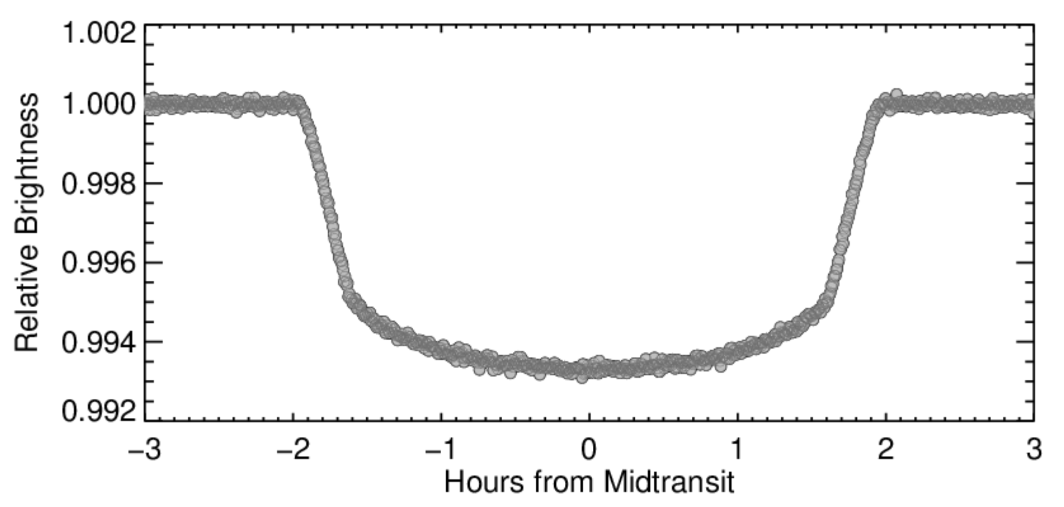
$$= 7980 + 2800\left(\ln 2 + \frac{17}{250}\right)$$

$$= \mathbf{1.01 \times 10^4}\ \boldsymbol{pc}$$ **3m**

(Q11) HAT-P-7 **[9 marks]**

HAT-P-7 is a F-type main sequence star that was discovered to host an exoplanet in 2008 by the Kepler Mission spacecraft. From SIMBAD and Welsh et al. (2010), the star has an apparent magnitude of $m_V = 10.46$, mass of $M = 1.53M_{sun}$, radius of $R = 1.98R_{sun}$, and a trigonometric parallax of $p = 0.00299''$. Shown below is the light curve of the system and an aggregated light curve of the *primary* transit, from Vanderburg. Calculate the radius and orbital radius of the exoplanet, and hence classify the exoplanet.





Solution:

Observe that the primary transits repeat with a period equal to the orbital period of the exoplanet. Hence its orbital period is

$$\begin{aligned}\langle P\rangle &= \frac{5.60-1.20}{2}\ days \\ &= 2.20\ days\end{aligned}$$

2m

As the gravitational force provides the centripetal force for the planet's orbit, for $M \gg m$,

$$\frac{mv^2}{a} = \frac{GMm}{a^2}$$

$$\because v = \frac{2\pi a}{\langle P \rangle} \Rightarrow \frac{4\pi a^2}{a\langle P \rangle^2} = \frac{GM}{a^2}$$

$$a^3 = \frac{GM\langle P \rangle^2}{4\pi^2}$$

$$a = \sqrt[3]{\frac{GM\langle P \rangle^2}{4\pi^2}}$$

$$= \sqrt[3]{\frac{(6.674 \times 10^{-11})(1.53 \times 1.989 \times 10^{30})(2.20 \times 24 \times 60^2)^2}{4\pi^2}}$$

$$= 5.71 \times 10^9\ m = \mathbf{0.0381\ AU}$$

3m

From the ratio of the brightness of the minima of the primary transit to the non-transit brightness, assuming the exoplanet radiates negligibly, we can calculate the physical radius of the exoplanet r to be

$$\frac{\pi R^2 - \pi r^2}{\pi R^2} = \frac{0.99325}{1.000}$$

$$1 - \left(\frac{r}{R}\right)^2 = 0.99325$$

$$\Rightarrow r = R\sqrt{1 - 0.99325}$$

$$= (1.98 \times 6.957 \times 10^8)\sqrt{1 - 0.99325}$$

$$= 1.13 \times 10^8\ m = \mathbf{1.62\ R_{Jup}}$$

3m

The exoplanet is a **hot Jupiter** given its small orbital radius and large radius. **1m**

(Q12) Keeping time **[9 marks]**

The Global Positioning System (GPS) is a navigation system that provides geolocation and time information to a receiver on Earth, when there is unobstructed line of sight to four or more GPS satellites, that are in Medium Earth orbit at an orbital radius of $r_{MEO} = 26{,}600\ km$. However, the combined effects of special and general relativity cause the clocks onboard the satellites to run 38 μs faster per day than clocks on Earth, which was corrected for in the design of the GPS. Calculate the change in orbital radius of the satellite network required, Δr, to minimise the corrections needed due to these relativistic effects. The gravitational time dilation at a radius r from a large and slowly rotating body can be approximated by

$$t = t_\infty \sqrt{1 - \frac{2GM}{rc^2}}$$

where t is the time elapsed and t_∞ is the time elapsed at a large distance from the body. As these relativistic effects are small, you may use the binomial approximation

$$(1 \pm x)^\alpha = 1 \pm \alpha x$$

Solution:

Observe that for satellites in MEO, SR time dilation results in time elapsing slower relative to a stationary ground observer, whereas GR time dilation results in time elapsing faster (for the satellite is in a weaker gravitational field further from Earth).

For a satellite in a circular orbit at some orbital radius r',

$$v = \sqrt{\frac{GM}{r'}}$$

From special relativity, the time elapsed t' at speed v in a circular orbit relative to a stationary reference t is

$$t' = \gamma t = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$= \frac{t}{\sqrt{1 - \frac{GM}{r'c^2}}}$$ **2m**

Hence, using the binomial approximation, time elapses slower for satellite clocks in orbit around the Earth by a fraction Δt equal to

$$\Delta t_{SR} \approx \frac{t'}{t} - 1 = \frac{1}{\sqrt{1 - \frac{GM}{r'c^2}}} - 1$$

$$\approx 1 - \frac{GM}{2\mathrm{r}'\mathrm{c}^2} - 1$$

$$= \frac{GM}{2\mathrm{r}'\mathrm{c}^2}$$ **1m**

From general relativity, using the binomial approximation, the gravitational time dilation at orbital distance r relative to an observer at infinity is

$$\frac{t}{t_\infty} \approx 1 - \frac{GM}{rc^2}$$ **1m**

Then, the difference is gravitational time dilation between an observer on Earth (with $r = r_E$) and a satellite in MEO at orbital radius $r = r'$ is

$$\Delta t_{GR} \approx \frac{t_E}{t_\infty} - \frac{t'}{t_\infty}$$
$$= \left(1 - \frac{GM}{r_E c^2}\right) - \left(1 - \frac{GM}{r' c^2}\right)$$
$$= \frac{GM}{c^2}\left(\frac{1}{r_E} - \frac{1}{r'}\right)$$

2m

To minimise the corrections needed, the magnitude of the two effects should be equal, i.e. $\Delta t_{SR} = \Delta t_{GR}$,

$$\frac{GM}{2\mathrm{r}'\mathrm{c}^2} = \frac{GM}{c^2}\left(\frac{1}{r_E} - \frac{1}{r'}\right)$$
$$\frac{1}{r'} = 2\left(\frac{1}{r_E} - \frac{1}{r'}\right)$$

1m

$$\frac{1}{r'}(1 + 2) = 2\left(\frac{1}{r_E}\right)$$
$$\frac{3}{r'} = \frac{2}{r_E}$$
$$r' = \frac{3}{2} r_E$$

1m

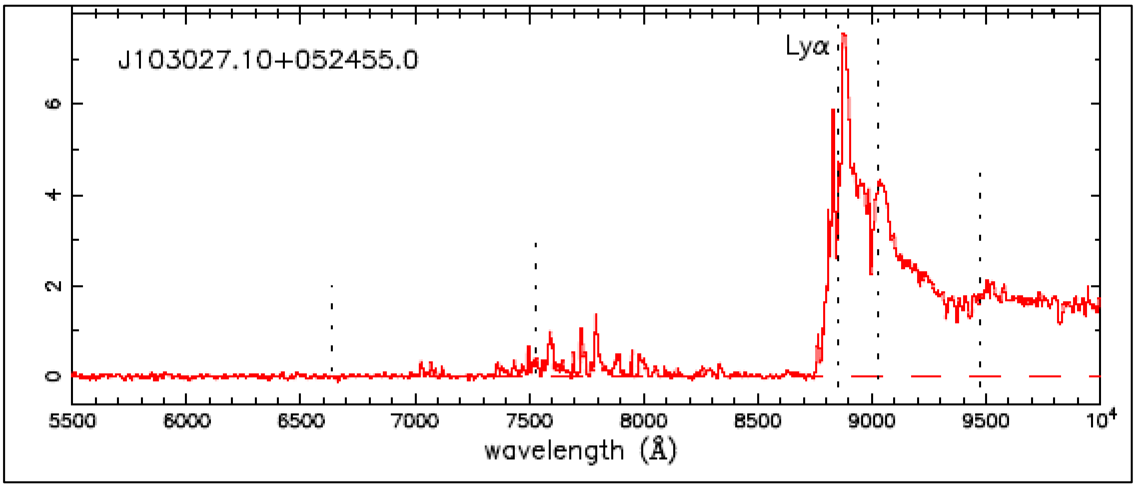
Then the change in orbital radius required is

$$\Delta r = r' - r_{MEO}$$
$$= \frac{3}{2}(6.371 \times 10^6\ m) - (2.66 \times 10^7\ m)$$
$$= \boldsymbol{-1.70 \times 10^7\ m}$$

1m

(Q13) The Gunn-Peterson trough **[9 marks]**

The Gunn-Peterson trough is a feature observed in the spectra of distant quasars due to the presence of relatively opaque neutral hydrogen in the Intergalactic Medium (IGM). The trough was first identified by Becker et al. (2001) from the suppression of electromagnetic radiation with wavelengths shorter than that of the Lyman-alpha line, at the redshift where the light was emitted. Shown below is the spectrum of the quasar in which the Gunn-Peterson trough was first identified in. Given that the rest wavelength of the Lyman-alpha line $\lambda_{Ly-\alpha} = 121.6\ nm$, calculate the scale factor a at the time the light was emitted. You may assume a matter-dominated Universe, for which the scale factor $a \propto t^{2/3}$.



Solution:

From the spectrum, the Lyman-alpha line has been redshifted to

$$\lambda_{obs} = 8850\text{Å} = 885\ nm$$ **2m**

The redshift factor z is then

$$1 + z = \frac{\lambda_{obs}}{\lambda_{rest}}$$

$$\Rightarrow z = \frac{\lambda_{obs}}{\lambda_{rest}} - 1$$

$$= \frac{8850\text{Å}}{1216\text{Å}} - 1$$

$$= 6.28$$ **3m**

From general relativity, for a homogeneous and isotropic Universe, deriving from the geodesic for a light wave, cosmological redshift is related to the redshift factor z and the scale factor a by

$$1 + z = \frac{a_{now}}{a_{then}}$$ **2m**

$$\because a_{now} \equiv 1 \Rightarrow a_{then} = \frac{1}{1+z}$$

$$= \frac{1}{1 + 6.28}$$

$$= \mathbf{0.137}$$ **4m**

Part C

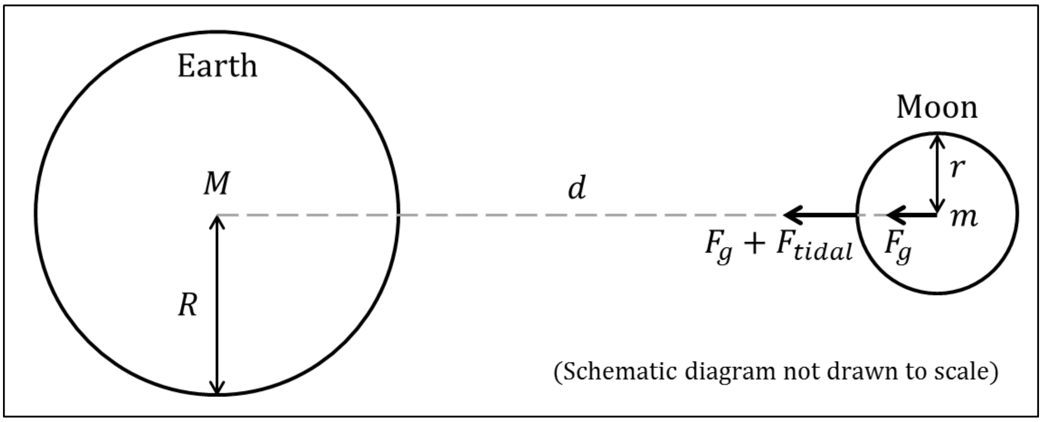
(Q14) The Earth and the Moon **[34 marks]**

Tidal forces are apparent forces that occur due to a gradient in the gravitational field across an orbiting body. The Earth-Moon system is the best studied case of tidal forces between two bodies; the Moon has sufficient mass and orbits sufficiently close to raise appreciable lunar tides on Earth.

(a) Consider the Earth, of mass M and radius R, and the Moon, of mass m and radius r, separated by a distance d. Show that the magnitude of the tidal force $|F_{tidal}|$ on the Earth from the Moon along the Earth-Moon axis can be approximated to

$$|F_{tidal}| \approx \frac{kMmr}{d^3}$$

where k is a constant to be expressed in other constants. You may use the binomial approximation $(1 \pm x)^a \approx 1 \pm ax$ for small x. [4]



Solution:

The magnitude of the tidal force is

$$|F_{tidal}| = F_g' - F_g$$

$$= \frac{GMm}{(d-r)^2} - \frac{GMm}{d^2}$$ **1m**

$$= \frac{GMm}{d^2}\left(\frac{1}{\left(1-\frac{r}{d}\right)^2} - 1\right) \approx \frac{GMm}{d^2}\left(1 + \frac{2r}{d} - 1\right)$$ **2m**

$$= \frac{GMm}{d^2}\left(\frac{2r}{d}\right)$$

$$= \frac{\boldsymbol{2GMmr}}{\boldsymbol{d^3}} \ (\boldsymbol{hence\ shown\ where\ k = 2G})$$ **1m**

However, as Earth's rotation is much faster than the orbital motion of the Moon, the tidal bulge is dragged ahead of the position directly under the Moon. This creates a torque between the Earth and the Moon, which increases the orbital semi-major axis of the Moon at the expense of the rotation of the Earth, a phenomenon known as *tidal acceleration*.

(b) Munk et al. (1998) estimates that the total tidal dissipation of energy by tidal friction averages about $|P_{tidal}| \approx 3.75\,TW$. Calculate the present rate of increase of the Earth's rotational period Δt_{Earth} **per year**, given that from Lambeck (1980), the moment of inertia of Earth along its polar axis, $I_{Earth} = 8.01 \times 10^{37}\ kg\ m^2$. [6]

Solution:

The rotational kinetic energy of the Earth with angular velocity ω is

$$\begin{aligned} K &= \frac{1}{2} I_{Earth} \omega^2 \\ &= \frac{1}{2} I_{Earth} \left(\frac{2\pi}{t_{Earth}}\right)^2 \\ &= \frac{2\pi^2 I_{Earth}}{t_{Earth}^2} \end{aligned}$$

2m

The present rate of increase of the rotational period is then

$$\begin{aligned} |P_{tidal}| &= \Delta K \\ &= K_i - K_f \\ &= 2\pi^2 I_{Earth} \left(\frac{1}{t_{Earth}^2} - \frac{1}{(t_{Earth} + \Delta t_{\mathrm{Earth}})^2}\right) \\ &= \frac{2\pi^2 I_{Earth}}{t_{Earth}^2} \left(1 - \frac{1}{\left(1 + \frac{\Delta \mathrm{t_{Earth}}}{t_{Earth}}\right)^2}\right) \\ &\approx \frac{2\pi^2 I_{Earth}}{t_{Earth}^2} \left(1 - \left(1 - \frac{2\Delta t_{Earth}}{t_{Earth}}\right)\right) \\ &= \frac{4\pi^2 I_{Earth}}{t_{Earth}^3} \Delta t_{Earth} \end{aligned}$$

2m

$$\begin{aligned} \Delta t_{Earth} &= \frac{t_{Earth}^3 |P_{tidal}|}{4\pi^2 I_{Earth}} \\ &= \frac{(23.935 \times 24 \times 60^2\ s)^3 (3.75 \times 10^{12}\ kg\ m^2\ s^{-3})}{4\pi^2 (8.01 \times 10^{37}\ kg\ m^2)} \\ &= 7.587 \times 10^{-13}\ s\ s^{-1} \\ &= \mathbf{2.394 \times 10^{-5}\ s\ yr^{-1}} \end{aligned}$$

1m

1m

Lunar laser ranging experiments with mirrors left on the Moon during the Apollo missions have confirmed the recession of the Moon in its orbit. From 1970 to 2002, Williams et. al (2002) gives the average rate of recession of the semi-major axis of the Moon **per year** as $\Delta a_{Moon} = 38.0\ mm/yr$.

(c) However, only a small fraction *f* of the total tidal power dissipated by the Earth is transferred to the orbit of the Moon; the bulk is dissipated as heat by tidal friction in the oceans and their interactions with the Earth's crust. Calculate this fraction *f*. [6]

Solution:

The orbital energy of the Moon is

$$T = -\frac{GMm}{2a_{Moon}}$$ **1m**

Then, the increase in the orbital energy of the Moon is

$$\begin{aligned}\Delta T &= T_f - T_i \\ &= -\frac{GMm}{2(a_{Moon} + \Delta a_{Moon})} + \frac{GMm}{2a_{Moon}} \\ &= -\frac{GMm}{2}\left(\frac{1}{a_{Moon} + \Delta a_{Moon}} - \frac{1}{a_{Moon}}\right) \\ &\approx -\frac{GMm}{2a_{Moon}}\left(\frac{1}{1 + \frac{\Delta a_{Moon}}{a_{Moon}}} - 1\right) \\ &= -\frac{GMm}{2a_{Moon}}\left(1 - \frac{\Delta a_{Moon}}{a_{Moon}} - 1\right) \\ &= \frac{GMm\Delta a_{Moon}}{2a_{Moon}^2}\end{aligned}$$ **1m** **2m**

Therefore, the fraction of the total tidal power dissipated by the Earth <u>in a year</u> that is transferred to the Moon is

$$\begin{aligned}f &= \frac{\Delta T}{T_{year}|P_{tidal}|} \\ &= \frac{GMm\Delta a_{Moon}}{2a_{Moon}^2 T_{year}|P_{tidal}|} \\ &= \frac{(6.674 \times 10^{-11})(5.972 \times 10^{24})(7.347 \times 10^{22})(38.0 \times 10^{-3})}{2(3.844 \times 10^{8})^2(365.26 \times 24 * 60^2)(3.75 \times 10^{12})} \\ &= \mathbf{0.0318\ i.e.\ 3.18}\%\end{aligned}$$ **1m** **1m**

As the Moon recedes in its orbit, it is expected that its angular diameter would eventually be too small to totally occlude the Sun as viewed from Earth during eclipses, even after considering the eccentricity of the Earth's and Moon's orbits.

(d) Calculate the **maximal time** from now T_{max}, in years, to when total solar eclipses will no longer be visible from Earth, for the trivial case where

(1) the rate of recession Δa_{Moon},
(2) the eccentricity of the Earth's and the Moon's orbits, and
(3) the diameter of the Sun

does not change appreciably over the timescale T_{max}. [8]

Solution:

The semi-major axis of the Moon after a time T in years, is

$$a' = a_{Moon} + T\Delta a_{Moon}$$ **1m**

For total solar eclipses to be no longer visible from Earth, consider the extremum case, where the angular size of the Moon at perigee $\theta_{Moon,p}$ is insufficient to occlude the angular size of the Sun at aphelion $\theta_{Sun,a}$, i.e. **1m**

$$\theta_{Moon,p} < \theta_{Sun,a}$$

$$\frac{2R_{Moon}}{d'_{Moon,p}} < \frac{2R_{Sun}}{d_{Sun,a}}$$ **1m**

$$\Rightarrow \frac{R_{Moon}}{a'(1-\epsilon_{Moon})} < \frac{R_{Sun}}{a_{Earth}(1+\epsilon_{Earth})}$$ **2m**

Rearranging and solving for T_{max},

$$a' > \frac{R_{Moon}a_{Earth}(1+\epsilon_{Earth})}{R_{Sun}(1-\epsilon_{Moon})}$$

$$a_{Moon} + T\Delta a_{Moon} > \frac{R_{Moon}a_{Earth}(1+\epsilon_{Earth})}{R_{Sun}(1-\epsilon_{Moon})}$$

$$T\Delta a_{Moon} > \frac{R_{Moon}a_{Earth}(1+\epsilon_{Earth})}{R_{Sun}(1-\epsilon_{Moon})} - a_{Moon}$$ **2m**

$$\Rightarrow T_{max} = \frac{\frac{R_{Moon}a_{Earth}(1+\epsilon_{Earth})}{R_{Sun}(1-\epsilon_{Moon})} - a_{Moon}}{\Delta a_{Moon}}$$

$$= \frac{\frac{(1.737\times 10^{6}\ m)(1.496\times 10^{11}\ m)(1+0.0167)}{(6.957\times 10^{8}\ m)(1-0.0549)} - 3.844\times 10^{8}\ m}{(38.0\times 10^{-3}\ m\ yr^{-1})}$$

$$= \mathbf{4.58\times 10^{8}\ yrs, i.e. 458\ million\ years}$$ **1m**

The magnitude of the tidal power $|P_{lunar}|$ dissipated from lunar tides can be modelled by a power law of the form

$$|P_{lunar}| = kd^{\alpha}$$

where α is some real number.

(e) Suggest a reasonable value for α, and explain qualitatively your reasons for your suggested value. Hence, comment on assumption (1). [3]

Solution:

A thorough calculation[1] gives $\tau = kd^{-6}$, and since $P = \tau\omega$, thus

$$P = (kd^{-6})\omega = k'd^{-6}$$

i.e. $\boldsymbol{\alpha = -6}$. **1m**

Answers with $\boldsymbol{\alpha \leq -1}$ score full credit.

From part (a), we observe that as the Moon's orbit is raised, the tidal force it exerts on the Earth decreases. Hence, we expect that the magnitude of the tidal power should likewise decrease, implying the exponent α is negative. **1m**

For a negative exponent α, this implies that the rate of recession of the Moon's orbit is likely to decrease with increasing semi-major axis of the Moon, i.e. Assumption 1 is **unlikely to hold true over $\boldsymbol{T_{max}}$**. **1m**

More rigorously, the tidal torque between the Earth and the Moon arises due to a mismatch between the angular velocity of the Earth's rotation and the Moon's orbital angular velocity,

$$\Delta\omega = \omega_{Earth} - \omega_{Moon}$$

As the Moon's orbit is raised, its angular velocity decreases. However, the slowing of Earth's rotation also reduces its angular velocity, but the magnitude of its effect is about an order of magnitude greater (recall that only a small fraction of the tidal power dissipated is transferred to the Moon's orbit). Hence, we expect the tidal torque and hence tidal power to correspondingly decrease with increasing orbital semi-major axis of the Moon, i.e. the exponent α is negative.

[1] Please refer to equation (6.90) of Introduction to Celestial Mechanics, from the University of Texas at Austin at http://farside.ph.utexas.edu/teaching/celestial/Celestial/node54.html#e6xx66

(f) Calculate the ratio of the tidal force on the Moon from the Earth at perigee to that at apogee. Hence, comment on assumption (2). [4]

Solution:

From part (a), the magnitude of the tidal force exerted on the Moon by the Earth is

$$|F_{tidal}| \approx \frac{2GMmr}{d^3}$$

Then, the ratio of the tidal force on the Moon at perigee to apogee is

$$\frac{|F_{tidal}|_p}{|F_{tidal}|_a} = \left(\frac{d_{apogee}}{d_{perigee}}\right)^3$$ **1m**

$$= \left(\frac{a_{Moon}(1+\epsilon_{Moon})}{a_{Moon}(1-\epsilon_{Moon})}\right)^3$$ **1m**

$$= \left(\frac{1+0.0549}{1-0.0549}\right)^3$$

$$= \mathbf{1.39}$$ **1m**

Assumption 2 is **unlikely to hold true over long time periods** as the effect of *tidal circularisation* is likely to reduce the eccentricity of the Moon's orbit. **1m** The tides raised by the Moon at perigee are greater than those at apogee. Hence, the perigee of the Moon will be raised more than the apogee due to the larger tidal power dissipation there. This will manifest as a reduction in the eccentricity of the orbit over long timescales.

(g) From your knowledge of stellar evolution, comment qualitatively on assumption (3). [1]

Solution:

Assumption 3 is **highly unlikely to hold true** as the Sun's radius increases with its luminosity on the main sequence, as it evolves towards the early stages of a red giant. **1m**

(h) From your answers in parts (e), (f) and (g), comment on how your calculated estimate for the trivial case in part (d) will differ from that produced by a more thorough calculation and simulation. [2]

Solution:

All three assumptions are **highly simplifying** and thus the accuracy of the calculated estimate from the trivial case is **minimal**. **1m**

We have to compare the effects from correcting for each of the three assumptions, to conclude if a more thorough calculation will have a larger or smaller T_{max}.

Assumption 1 will lead to a decrease in the rate of recession of the Moon, thus increasing T_{max} by slowing the shrinking of the Moon's apparent angular size.

Assumption 2 will lead to a small decrease in T_{max} as the variance in the Moon's angular size decreases; for the trivial case we are only considering the Moon at its largest possible angular size, at perigee.

Assumption 3 will lead to a decrease in T_{max} as the Sun's angular size increases with more time spent on the main sequence.

Comparing the three effects, assumption 1 introduces the greatest error but is opposite in effect to assumptions 2 and 3 – hence it is reasonable to conclude T_{max} for a thorough calculation will be **longer**. **1m**

Answers showing logical reasoning or evidence of error analysis gain full credit.

Part D

(Q15) The stellar mass-luminosity relation **[40 marks]**

The mass-luminosity relation for main sequence stars was first proposed in a paper by astronomer Jakob Karl Ernst Halm in 1911. This empirical relation has the functional form

$$L \propto M^{\alpha} \Rightarrow \frac{L}{L_{Sun}} = k\left(\frac{M}{M_{Sun}}\right)^{\alpha}$$

where L is the luminosity of the star, M is the mass of the star, L_{Sun} is the solar luminosity, M_{Sun} is the solar mass, k is a proportionality constant that depends on the mass of the star, and $\beta \leq \alpha < 6$ is the exponent of the power law.

(a) Suggest a value for β, and briefly justify your value with reference to stellar structure and processes in main-sequence stars. [2]

Solution:

Consider that, as a main sequence (MS) star increases in mass, the inward gravitational pressure on the core from the overlying weight of the star's envelope increases. The **outward radiation pressure (star's luminosity) thus needs to increase correspondingly for hydrostatic equilibrium** to be maintained, implying that $\boldsymbol{\alpha = 1}$. **1m** **1m**

For $\alpha < 1$, this would imply the star's luminosity (and hence radiation pressure in the core) decreases with increasing mass. Such stars do not exist on the MS and will fail to be in hydrostatic equilibrium.

Note that for MS stars, the reactions for the fusion of hydrogen to helium are highly sensitive to temperature; in reality, a small increase in gravitational compression produces a significant increase in a star's luminosity.

(b) An approximate value of α can be derived from basic physics and simplifying assumptions. Consider an ideal spherical star of mass M and radius R, with average density $\bar{\rho}$ and average particle mass $\bar{m}$. For hydrostatic equilibrium, the average pressure $\bar{P}$ is

$$\bar{P} = -\frac{E_g}{3V}$$

where $E_g = -\frac{3GM^2}{5R}$ is the gravitational potential energy of a spherical mass distribution of volume V. Using the Stefan-Boltzmann law and ideal gas law in the form

$$PV = Nk_BT$$

where N is the number of gas particles and k_B the Boltzmann constant, show that $\alpha = 10/3$. [5]

Solution:

Substituting the volume V and gravitational potential energy E_g of the spherical star into the equation for average pressure $\bar{P}$ in hydrostatic equilibrium, we have

$$\bar{P} = -\frac{-\frac{3GM^2}{5R}}{3V}$$
$$= \frac{3GM^2}{5RV}$$ **1m**

Applying the ideal gas law $PV = Nk_BT$ on the star as directed, and substituting the average particle mass $\bar{m}$ we have

$$\bar{P} = \frac{Nk_BT}{V}$$
$$\because N = \frac{M}{\bar{m}} \Rightarrow \bar{P} = \frac{M}{\bar{m}V}k_BT$$ **1m**

Equating the two above,

$$\frac{3GM^2}{5RV} = \frac{M}{\bar{m}V}k_BT$$
$$RT = \frac{3GM}{5k_B\bar{m}}$$ **1m**

Observing that

$$M = \bar{\rho}V = \frac{4}{3}\pi R^3\bar{\rho}$$
$$\Rightarrow R = \left(\frac{3M}{4\pi\bar{\rho}}\right)^{1/3}$$ **1m**

Substituting the temperature T and radius R into the Stefan-Boltzmann equation, we get

$$L = 4\pi R^2\sigma T^4 = \frac{4\pi\sigma(RT)^4}{R^2}$$
$$= \frac{4\pi\sigma\left(\frac{3GM}{5k_B\bar{m}}\right)^4}{\left(\frac{3M}{4\pi\bar{\rho}}\right)^{2/3}}$$
$$= \frac{(3^{10/3})(4^{5/3})\pi^{5/3}\sigma G^4M^{10/3}\bar{\rho}^{2/3}}{(5^4)k_B^4\bar{m}^4}$$ **1m**
$$\Rightarrow L \propto M^{10/3}$$

Hence shown that $\alpha = 10/3$.

In reality, surveys of nearby stars and binaries on the main sequence show that the approximate mass-luminosity relation of $L \propto M^{10/3}$ deviates significantly from observed values for low- and high-mass stars.

Eker et al. (2015) proposes that a piecewise mass-luminosity relation provides a better fit for stars from different mass ranges – due to differences in internal structure and energy production rates per stellar mass M/L. Data of 55 stars within the mass range $0.38M_{Sun} \leq M < 23M_{Sun}$ are provided on Page 13.

(c) Plot $\lg\left(\frac{M}{L}\right)$ against M for $M < 7M_{Sun}$ to obtain **Graph 1A**. You should obtain a graph with two distinct breakpoints M_1, M_2 where $M_1 < M_2$, that divide the data into three linear distributions. Mark out M_1, M_2 on **Graph 1A** and indicate the corresponding mass boundaries. [6]

Solution:

Plots **all 42** relevant data points (from S/N 290 to S/N 244) to obtain **Graph 1A** (see attached spreadsheet). Reading off from **Graph 1A**, the breakpoints (marked out) are at **2m**

$M_1 = 1.10M_{Sun}$ **2m**

$M_2 = 2.70M_{Sun}$ **2m**

Award full credit for M_1 within $0.1M_{Sun}$ (inclusive) of the above value, and half credit if within $0.2M_{Sun}$ (inclusive). Award full credit for M_2 within $0.2M_{Sun}$ (inclusive) of the above value, and half credit if within $0.3M_{Sun}$ (inclusive). Award no credit for that breakpoint if it is not marked out on the graph.

(d) Plot $\lg\left(\frac{M}{L}\right)$ against M for masses $M > 3M_{Sun}$ to obtain **Graph 1B**. You should obtain a graph with an additional distinct breakpoint M_3 that divides the remaining data into two linear distributions, for a total of four different linear distributions. [3]

Solution:

Plots **all 23** relevant data points (from S/N 160 to S/N 143) to obtain **Graph 1B** (see attached spreadsheet). Reading off from **Graph 1B**, the breakpoint **1m**

$M_3 = 8.0M_{Sun}$

2m

Award full credit for M_3 within $0.5M_{Sun}$ (inclusive) of the above value, and half credit within $1.0M_{Sun}$ (inclusive). Award no credit if the M_3 is not marked out on the graph.

The piecewise mass-luminosity relation for these four mass distributions of stars (very low-, low-, intermediate-, high-) can be expressed as

$$\frac{L}{L_{Sun}} = \begin{cases} 1.062\left(\frac{M}{M_{Sun}}\right)^{4.841}, & 0.38M_{Sun} < M \leq M_1 \\ k_2\left(\frac{M}{M_{Sun}}\right)^{q}, & M_1 < M \leq M_2 \\ k_3\left(\frac{M}{M_{Sun}}\right)^{r}, & M_2 < M \leq M_3 \\ k_4\left(\frac{M}{M_{Sun}}\right)^{s}, & M_3 < M < 23M_{Sun} \end{cases}$$

where k_2, k_3, k_4 are proportionality constants to be found, and M_1, M_2, M_3 are the mass boundaries that divide the data into four mass ranges.

(e) Plot three linear graphs to obtain the values of k_2, k_3, k_4 and q, r, s. Label your graphs **Graph 2**, **Graph 3**, **Graph 4** in the order of the increasing mass ranges. [14]

Solution:

Linearising the general mass-luminosity relation to fit the data provided,

$$\lg\left(\frac{L}{L_{Sun}}\right) = \alpha \lg\left(\frac{M}{M_{Sun}}\right) + \lg k$$ **2m**

With the values from **Graph 1A** and **Graph 1B**, plot a graph for each of the following mass distributions (see attached spreadsheet for plots), adhering to the boundaries marked out in (b) and (c),

(1) $1.1M_{Sun} < M \leq 2.7M_{Sun} - \boldsymbol{Graph\ 2}$ **2m**
(2) $2.7M_{Sun} < M \leq 8.0M_{Sun} - \boldsymbol{Graph\ 3}$ **2m**
(3) $8.0M_{Sun} < M \leq 23M_{Sun} - \boldsymbol{Graph\ 4}$ **2m**

Award full credit (2m) for each of the above graphs correctly plotted with points belonging to that mass distribution. Award half credit for that graph if erroneous points are included that do not belong to that mass distribution given the breakpoints found in (b) and (c).

Plotting each of the above three graphs, we obtain the following values

$\lg k_2 = 0.0194 \Rightarrow k_2 = 1.046, q = 4.262$ **2m**
$\lg k_3 = 0.108 \Rightarrow k_3 = 1.282, r = 3.971$ **2m**
$\lg k_4 = 1.526 \Rightarrow k_4 = 33.57, s = 2.539$ **2m**

Award full credit (1m) for each of the values $\boldsymbol{k_2, k_3, k_4, q, r, s}$ calculated correctly, within 25% of the respective values above.

(f) Describe the trend in the exponents of the mass-luminosity relation with increasing mass. [1]

Solution:

From $0.38M_{Sun} \leq M < 23M_{Sun}$, as $M \to 23M_{Sun}$, the **exponent $\boldsymbol{\alpha}$ decreases**. **1m**

(g) Using the appropriate relation, calculate the expected absolute V-band magnitude M_V of the star *Fomalhaut*, given its mass $M = 1.92M_{Sun}$ as measured by Mamajek (2012) through astrometry. The bolometric correction for *Fomalhaut*, of spectral type A3V, is $BC = -0.07\ mag.$ [6]

Solution:

Fomalhaut, with mass $1.1M_{Sun} < M = 1.92M_{Sun} \leq 2.7M_{Sun}$ follows the following mass-luminosity relation

$$\frac{L}{L_{Sun}} = k_2\left(\frac{M}{M_{Sun}}\right)^p$$

$$\Rightarrow \frac{L}{L_{Sun}} = 10^{0.0194}\left(\frac{M}{M_{Sun}}\right)^{4.262}$$

Substituting $M = 1.92M_{Sun}$ and rearranging, *Fomalhaut's* luminosity L is

$$\begin{aligned} L &= 10^{0.0194}(1.92)^{4.262}L_{Sun} \\ &= 1.046(1.92)^{4.262}L_{Sun} \\ &= 16.86\ L_{Sun} \end{aligned}$$

2m

Then the absolute bolometric magnitude of *Fomalhaut* M_{bol} is

$$\begin{aligned} M_{bol} &= M_{Sun} - 2.5\lg\left(\frac{L}{L_{Sun}}\right) \\ &= 4.83 - 2.5\lg(16.86) \\ &= 1.763 \end{aligned}$$

2m

From the definition of the bolometric magnitude,

$$\begin{aligned} BC &= M_{bol} - M_V \\ &\Rightarrow M_V = M_{bol} - BC \\ &= 1.763 - (-0.07) \\ &= \mathbf{1.83} \end{aligned}$$

2m

For reference, the actual observed absolute V-band magnitude of *Fomalhaut* is 1.75.

The trend identified in part (f) however, does not hold for stars with very low masses of $M \leq 0.38M_{Sun}$. The mass-luminosity relation for such stars is approximately

$$\frac{L}{L_{Sun}} \approx 0.23\left(\frac{\mathrm{M}}{\mathrm{M_{Sun}}}\right)^{2.3}$$

(g) With reference to the stellar structure and processes in very low-mass stars, suggest a reason for the small exponent in its mass-luminosity relation. [3]

Solution:

In very low-mass stars with $M \leq 0.38M_{Sun}$, **<u>gravitational compression is insufficient to produce high interior temperatures</u>**. As a result, hydrogen and other metals are only partially ionised, and in the cool outer regions molecular hydrogen H_2 and H^- ions are formed appreciably. Bound-free absorption (dissociation and ionisation of molecules/atoms) and H^- absorption from these sources contribute to the high opacities found in their envelopes. **1m**

As a result, **<u>radiative transfer is inhibited, causing very low-mass stars to be entirely convective throughout their volume</u>** as the temperature gradient is sufficiently steep for convection to be more efficient at energy transport than radiation. **1m**

Hence, in these stars, **<u>an increase in gravitational compression and subsequently core temperatures produces an increase in dissociation and ionisation of the overlying layers</u>**, reducing the exponential increase in luminosity with stellar mass, explaining the much smaller exponent α. **1m**

Part P(A)

(P1) The Andromeda-Milky Way merger **[2 marks]**

From measurements of the Andromeda Galaxy's blueshift and transverse velocity, scientists predict that the Milky Way and Andromeda Galaxy (M31) will gravitationally interact and merge in about 4 billion years. Given that M31 and the Milky Way are the largest and second-largest galaxies in the Local Group respectively, indicate in the box the option that describes the most likely classification of the resultant galaxy after the merger is complete.

Answer: A, Elliptical galaxy (type *E*) **2m**

Computer simulations show that the resultant galaxy from the collision of the Milky Way and *M31* is likely to be an elliptical galaxy of type *E*. This is due to the head-on nature of the collision, the relative sizes of the two galaxies and the differing inclination of M31's galactic disk relative to the Milky Way's.

Observations of distant galaxies at different evolutionary stages also strongly support that elliptical galaxies are formed from mergers of smaller galaxies, as they are predominantly found in the gravity well at the centre of galaxy clusters with high rates of galactic interactions.

(P2) Sunrise in Singapore **[2 marks]**

The time of sunrise in Singapore remains relatively unchanged across the course of a year. Indicate in the box the option that is closest to the median time of sunrise in Singapore, in local time (hours and minutes).

Answer: C, 0700 **2m**

Singapore is very close to the Equator, where the Sun rises perpendicular to the horizon and thus half of the circle it traces out on the celestial sphere is above the horizon. Hence the Sun remains above the horizon for about 12 hours, rising at 0600h and setting at 1800h.

However, Singapore's latitude of $103°49'E$ is geographically offset from its timezone of UTC+08:00, by an angle

$$\Delta\theta = 8\left(\frac{360}{15}\right) - 104 \approx 16°$$

This corresponds to Singapore's local time being ahead of the sky by slightly more than one hour. Hence, the median time of sunrise is about one hour after 0600h, i.e. 0700h.

(P3) Seeking alignment **[2 marks]**

Indicate in the box the option that describes the most accurate method of polar alignment for an astrophotography setup.

Answer: D, Drift alignment **2m**

North alignment using a compass is the most inaccurate – it approximates the position of the celestial pole to the corresponding magnetic pole (in practice the local geomagnetic field), which are separated by about 4 degrees. This projection of the celestial pole to the horizon provides a very rough indicator of the azimuth of the celestial pole, without giving indication to its altitude. Hence, it is viable only for countries near the Equator where the celestial pole lies close enough to the horizon, with typical error of $\Delta\theta \approx 5°$.

Three-star alignment has moderate accuracy and is often used to refine rough north alignments from a compass. By sequentially centering three bright stars in the eyepiece of a telescope, a motorised mount can computationally construct a model for the local night sky and correct for small errors in the initial alignment of the mount head. The error associated varies with the expertise of the operator in centering the star, but is generally about $\Delta\theta \approx 2°$.

Alignment to the polar axis using a close reference star has moderate accuracy– for mounts with polar scopes, one simply has to align Polaris (in the northern hemisphere) or Sigma Octantis (in the southern hemisphere) to a predefined position in the reticle to centre onto the north celestial pole. This method provides both the azimuth and altitude of the celestial pole and has one order of magnitude less error, about $\Delta\theta \approx 0.5°$.

Drift alignment is used to further refine alignments from all of the above. Being the most tedious and time-consuming method of the four, it also provides the highest accuracy. It is achieved by computer software-aided iteration and correction to the alignment of the altitude and azimuth of the mount, to a desired level of accuracy, with accuracies exceeding $\Delta\theta < 0.1°$. This method is the gold standard for alignment and combined with an autoguider, is employed for extremely long exposures on the order of hours for a single exposure.

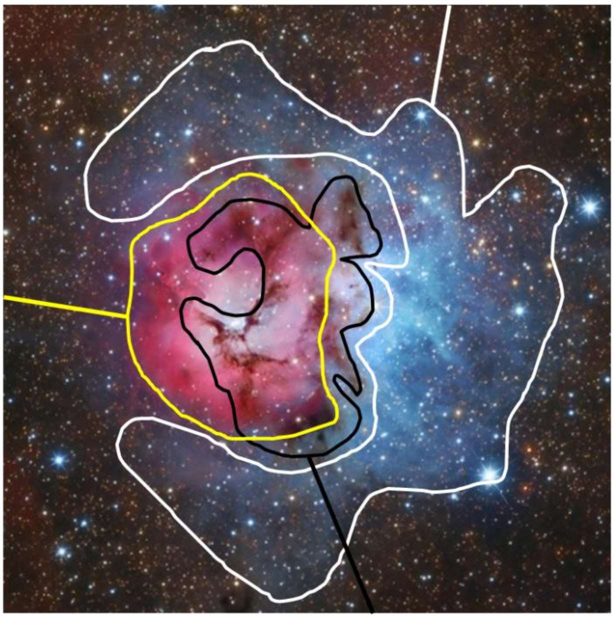
(P4) The Trifid Nebula **[2 marks]**

Messier 20, the *Trifid Nebula*, is popular amongst amateur astronomers as it contains an unusual combination of multiple types of nebulae. Tick in the adjacent box, all options describing a component nebulae found in M20.

Answer: H II region (emission nebulae), reflection nebulae, dark nebulae **2m**

M20 is a H II region associated with a star-forming region in the Scutum arm of the Milky Way. Hence, planetary nebulae and supernova remnant components are not found in it, as these are astronomical objects associated with the end-stages of stars.

An annotated photo of *M20* is shown below.



Credit:
Lorand Fenyes,
APOD NASA
(28 Aug 2014)

The emission component is due to emission at the Hα line from the ionised interstellar cloud (of atomic hydrogen) surrounding the cluster of newly-formed and hot stars. The absorption component arises due to the dense opaque dust clouds and lanes present near the centre of the interstellar cloud. The reflection component is due to preferential scattering of shorter wavelength light from the newly formed cluster off dust grains, illuminating the sparse sparse outer part of the interstellar cloud.

(P5) The opposition **[2 marks]**

Tick in the adjacent box, all options that describes a pair of Solar System bodies that can never be in opposition to each other to an observer on Earth.

Answer: None (i.e. all boxes left blank) **2m**

Most Solar System objects (including all of those in the list of options) orbit close to the ecliptic plane, thus opposition occurs during syzygy.

For two Solar System bodies to be in opposition, they must appear in opposite hemispheres of the celestial sphere.

The Sun and Saturn return to opposition every synodic period of Saturn, equal to 1.035 Earth sidereal years. This is also the typical definition of opposition, where a planet and the Sun's apparent geocentric longitudes differ by about 180^o.

Mars and Jupiter are both superior planets, and can be in opposition to each other when either planet is near conjunction with the Sun and the other planet is near opposition with the Sun.

Mercury is an inner planet and hence can only be in conjunction with the Sun. However, as the Moon orbits the Earth, it can come into opposition with Mercury when it is on the opposite hemisphere of the Earth, i.e. near full Moon.

Venus is an inner planet and hence can only be in conjunction with the Sun. However, it can come into opposition with Ceres when Ceres is near opposition with the Sun.

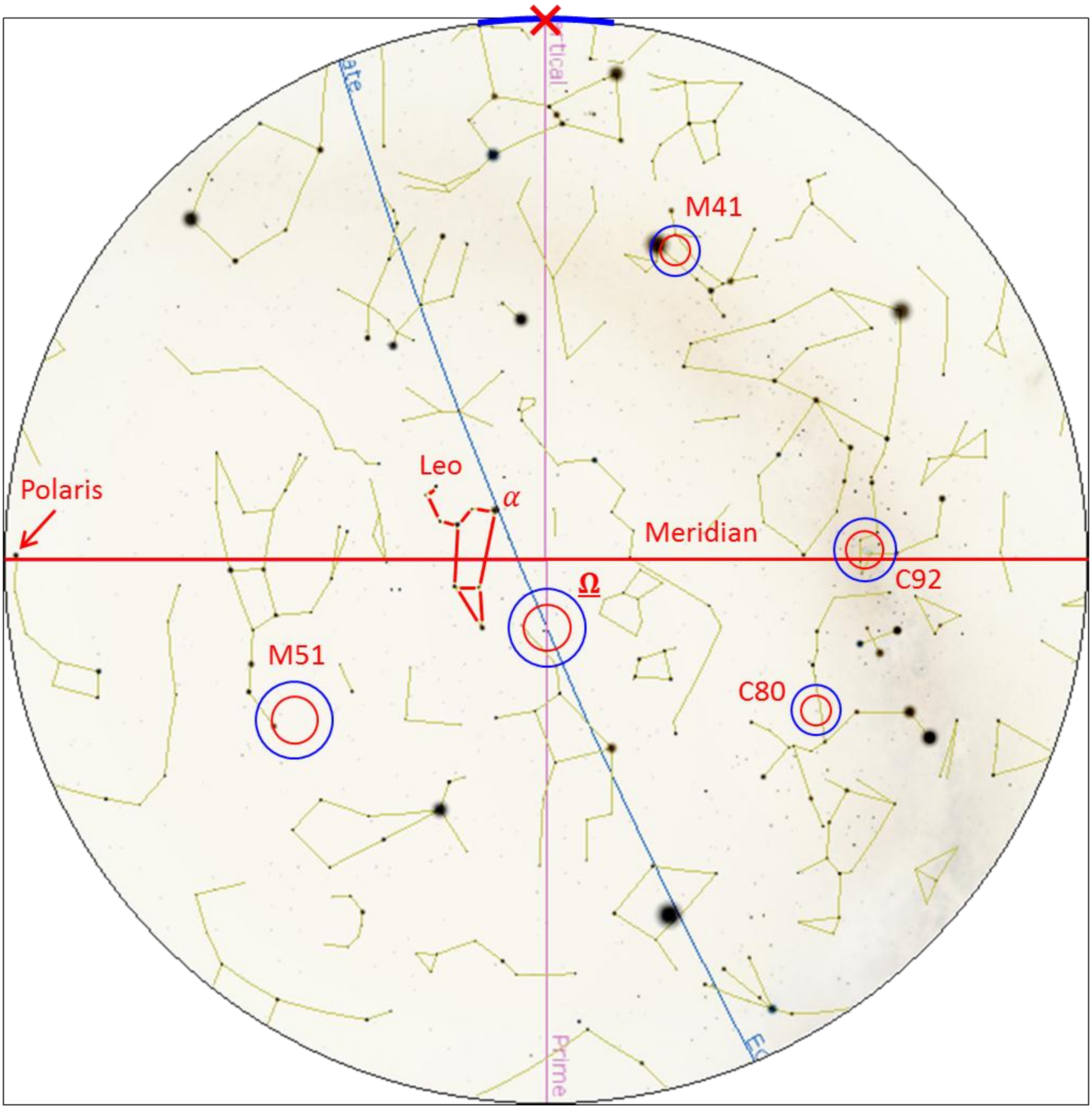
The Sun and the Moon come into opposition every full Moon – when this is exact, a lunar eclipse occurs.

Part P(B)

(P6) The night of Vernal Equinox **[28 marks]**

The star chart below shows the night sky at 0000h on the night following the Vernal Equinox (on 21 March 2018), from Singapore (UTC +08:00), at latitude $1° \, 17' N$ and longitude $103° \, 51' E$.

Complete the questions on the following page. Note that the size of stars and objects are scaled by their brightness in the night sky, with brighter objects appearing larger.



(a) Along the horizon (circumference of the star chart), mark out the approximate location of cardinal West, with a cross ×. [2]

Answer: See annotated star chart.

Award full credit if answer is within the red section, and half credit if within the blue section.

2m

(b) Trace out the local meridian with a solid arc, and label it M. [2]

Answer: See annotated star chart.

Award full credit (1m) for each of the following criteria:
(1) the endpoints of the meridian are diametrically opposite each other;
(2) a straight horizontal line from N to S (looking vertically up).

2m

(c) Trace out the constellation of *Leo* with solid lines connecting its stars. Label its alpha star α on the star chart. [4]

Answer: See annotated star chart.

Award full credit (2m) for all connections of Leo including the stars *Regulus*, *Denebola*, *Algieba* and *the Sickle of Leo*, without including any stars from other constellations. Award half credit (1m) if the connections include *Regulus*, *Denebola* and *Algieba*, but incorrectly includes a star from another constellation.

Award full credit (2m) for unambiguous identification of *Regulus*.

4m

(d) The pole star *Polaris*, *α Ursae Minoris*, is visible. Mark out the star with an arrow →. The tip of the arrow should point unambiguously at the star. [2]

Answer: See annotated star chart.

Award full credit (2m) for unambiguous identification of *Polaris*.

2m

(e) The following four deep sky objects (DSOs) are visible in the star chart. Mark out any three of these DSOs, each with a hollow circle O, and write that DSO's catalogue designation adjacent to it. The centre of the hollow circle will be taken as the position of that DSO. [9]

(i) M41, an open cluster
(ii) C80, the *ω Centauri Cluster*
(iii) C92, the *Eta Carinae Nebula*
(iv) M51, the *Whirlpool Galaxy*

Answer: See annotated star chart. **9m**

Award full credit (3m) for each DSO whose position is correctly identified within the red circle. Award partial credit (1m) for each DSO whose position is identified within the larger blue circle.

(f) Calculate the local hour angle of the first point of Libra. [5]

Solution:

By definition, the first point of Libra corresponds to the position of the descending node of the ecliptic relative to the celestial equator, and is approximately diametrically opposite to the first point of Aries, which has RA $\alpha = 0^h$. Hence the RA of the first point of Libra $\alpha' = 12^h$.

By definition, the right ascension of the Sun at local noon on the day of the vernal equinox is $RA = 0^h$. Then, the right ascension of the meridian at the following midnight is $RA = 12^h$. However, we have to correct for Singapore's time zone T being ahead of the local solar time so as to find the LST of the meridian at midnight local (civil) time, i.e.

$$\begin{aligned} LST_{0000} &= LST - \Delta\lambda \\ &= LST - \left(T - \frac{\lambda}{15}\right) \\ &= 12^h - 8^h + \left(\frac{103 + \frac{51}{60}}{15}\right)^h \\ &= 10^h 55^m \end{aligned}$$

Hence, from the local sidereal time LST_{0000} and right ascension RA_{Libra} of the first point of Libra, we can calculate its hour angle

$$\begin{aligned} LHA &= LST_{0000} - RA_{Libra} \\ &= 10^h 55^m - 12^h \\ &= \mathbf{-1^h 5^m \ OR \ 22^h 55^m} \end{aligned}$$

5m

Award full credit if answer is within 15^m of above and half credit if within 30^m of above, no credit otherwise.

(g) From your answer in (f), mark out the first point of Libra with a cross ×, and label it Ω. State the constellation whose boundaries the first point of Libra currently resides within. [4]

Answer: See annotated star chart. **2m**

From (f), observe that the first point of Libra has not crossed the local meridian given its negative hour angle. Hence, it is expected to be east of the local meridian, i.e. the bottom half of the star chart.

Recall that as the first point of Libra is the descending node (where the ecliptic intersects the celestial equator), then its declination is $\delta = 0°$. In Singapore, where the latitude is close to $\phi = 0°$, then the first point of Libra is expected to lie somewhere along the prime vertical (great circle connecting zenith, east and west).

Thus, the first point of Libra is expected to be trivially, about $1/6$ of the radius of the star chart down from its centre (in practice $\sim 1/8$ due to increasing distortion from the stereographic projection towards the horizon).

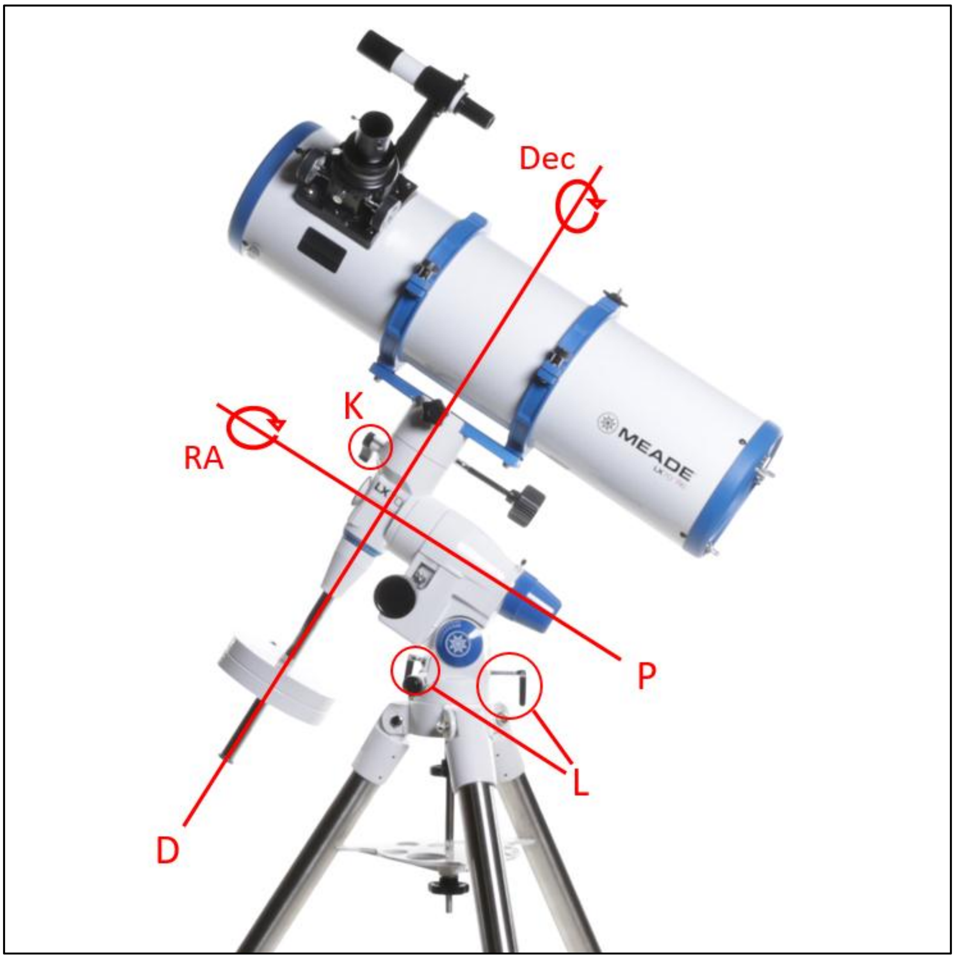
Award full credit (2m) if the first point of Libra is marked within the red circle, and half credit (1m) if within the blue circle.

Recall that due to axial precession, the first point of Libra shifts westward about one degree every 72 years; in the time since it was defined in 130 BC by Hipparchus, the first point of Libra has crossed from Libra into the neighbouring constellation of **Virgo** (it is located near the top of the "head" of Virgo). **2m**

Part P(C)

(P7) Orthogonal **[10 marks]**

Shown below is the manufacturer's image of the *6" f/5 Meade LX70 R6 Newtonian optical tube assembly* on the *Meade LX70 Equatorial Mount*.



(a) On the image, indicate the polar and declination axes of the telescope by drawing a straight line through each and labelling them P and D respectively. [4]

Answer: See annotated diagram. **4m**

Award full credit (2m) for each axis unambiguously identified. Award only half credit (1m) if the axis is not drawn straight with a ruler.

(b) On the image, indicate the directions of motion in RA and declination by drawing a circular arrow ↺ around the appropriate axes and labelling them RA and Dec respectively. [2]

Answer: See annotated diagram. **2m**

Award full credit (1m) for each direction of motion unambiguously identified around the correct axis, none otherwise.

(c) On the image, indicate the declination clutch locking knob by circling it and labelling it K. [2]

Answer: See annotated diagram. **2m**

Award full credit (2m) if the declination clutch locking knob is clearly identified, none otherwise.

(d) On the image, indicate the latitude adjustment knobs by circling them and labelling it L. [2]

Answer: See annotated diagram. **2m**

Award full credit (2m) if the latitude adjustment (setting) knob is clearly identified, none otherwise.

(P8) Colours **[12 marks]**

Shown below is a colour-inverted image of M57, the *Ring Nebula* in constellation *Lyra*, as imaged by the Hubble Space Telescope. At an astronomy retreat at a dark sky site, another astronomer asks you to help him image *M57*. In the space below, plan out a step-by-step procedure for imaging *M57*, including details of your setup, equipment used and processing to the final image, providing specification where necessary.

This question is marked by points fulfilled in the procedure – there are 11 marking points and 6 error reduction marking points.

Setup Procedure (max 7m)

(S1) Makes reference to levelling the tripod with a spirit level. **1m**

(S2) Makes reference to aligning the tripod approximately with a compass to either cardinal North or South. **1m**

(S3) Describes setting up a motorised German Equatorial Mount (GEM) atop a tripod, and then setting up an optical tube assembly (OTA) atop the mount. **1m**

(S4) Describes attachment of counterweight prior to telescope and all accessories. **This marking point is not awarded if the total weight of the OTA and attachments exceeds the rated weight capacity of the mount**.	**1m**
(S5) Describes attachment of accessories used (e.g. finder scope, autoguider if being used) and mounting of camera on the OTA (mentioning the respective adapter, e.g. T-ring and T-adapter for DSLRs).	**1m**
(S6) Details balancing of the declination axis followed by the RA axis, after attachment of all equipment parts.	**1m**
(S7) Describes briefly a method to achieve mount polar alignment (calibration), such as three-star alignment, but preferably polar alignment or drift alignment.	**1m**
(S8) Specifies a reasonable number (≥ 30) and length of exposures of *M57* (light frames) captured through the camera, with total length of exposures of at least 3 hours.	**1m**
(S9) Specifies the ISO setting for exposures made with DSLRs or the gain setting for exposures made with CCDs.	**1m**
<u>Choice of OTA and camera (max 3m)</u>	
(A1) Specifies the OTA's optical design (reflector – Newtonian, refractor – achromatic/apochromatic, catadioptric) and any two of the following three details: focal length, aperture, focal ratio. **Note that astrograph is not an optical design, and that this marking point is only awarded if the OTA fills the image sensor completely.**	**1m**
(A2) Specifies the design (DSLR or CCD) of the camera used and any two of the following three details: pixel dimensions, pixel array size, sensor size. **This marking point is only awarded if the choice of the OTA and camera are such that *M57* is not significantly oversampled or undersampled, specifically the resultant pixel scale should be between $1'' to\ 5''$ per pixel.**	**2m**
<u>Reduction of Systematic and Random Error (max 4m)</u>	
(E1) Describes reduction of thermal currents/turbulence in the OTA by allowing the telescope to equilibrate with the ambient temperature.	**1m**
(E2) Describes use of an autoguider (either attached to finder scope or as an off-axis guider) to continuously refine guiding by issuing adjustments to the mount's motors, reducing drifting of the object in the image.	**1m**

(E3) Specifies a DSLR or CCD with in-built thermoelectric cooling capability to reduce dark current in the exposure.	**1m**
(E4) Specifies the number and length of dark frames (both equal to that of the light frames) taken at the same ambient conditions to calibrate out thermal noise and read noise from the camera.	**1m**
(E5) Specifies the number and length of bias frames taken to correct for vignetting and imperfections from the optical train.	**1m**
(E6) Briefly describes post-processing of the RAW images in an astrophotography stacking program or image editor. **This marking point is only awarded if at least one of E4 or E5 are awarded and reference is made to use of those calibration frames**.	**1m**